Dynamics in Network Interaction Games

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Abstract

We study the convergence times of dynamics in games involving graphical relationships of players. Our model of local interaction games generalizes a variety of recently studied games in game theory and distributed computing. In a local interaction game each agent is a node embedded in a graph and plays the same 2-player game with each neighbor. He can choose his strategy only once and must apply his choice in each game he is involved in. This represents a fundamental model of decision making with local interaction and distributed control. Furthermore, we introduce a generalization called 2-type interaction games, in which one 2-player game is played on edges and possibly another game is played on non-edges. For the popular case with symmetric 2×2 games, we show that several dynamics converge in polynomial time. This includes arbitrary sequential better response dynamics, as well as concurrent dynamics resulting from a distributed protocol that does not rely on global knowledge. We supplement these results with an experimental comparison of sequential and concurrent dynamics.

1 Introduction

In this paper we examine convergence of dynamics in a fundamental model for distributed decision making with local interactions movtivated by distributed computer systems and social networks. We introduce two game-theoretic models, one a generalization of the other, that combine strategic interaction with the notion of graph-based locality. This extends a variety of game-theoretic settings that have been studied intensively in the literature. In our model of a *local interaction game* there is a graph G along with a 2-player symmetric game, Γ. Players are the nodes, and the graph models the local interaction possibilities. In particular, Γ is played along each edge of G, and each player plays the same strategy against each of their neighbors. The payoff of a player is simply the sum of the payoffs earned from playing each neighbor. Local interaction games are a basic framework to capture many different types of real-world phenomena, e.g., when a person is trying to coordinate with as many of his or her neighbors as possible. Similarly, the graph could encode antipathies and actors could strive to anti-coordinate with their neighbors. This is the general incentive in many computational resource sharing environments like channel assignment in wireless networks, where nodes try to choose a frequency that minimizes the number of spatially close nodes using the same frequency. It is thus not surprising that a large number of specific local interaction games have been studied in the literature [8, 28, 30, 34].

We also introduce a generalization of local interaction games called *2-type interaction games*. Intuitively, a 2-type interaction game is a graph where one 2-player symmetric game is played on the edges, and another 2-player symmetric game is played on the non-edges. Whereas local interaction games model the restricted interaction *possibilities* of players through the topology of the graph, 2-type interaction games also model different *types* of interactions that occur between players. This is a natural assumption when considering e.g. social networks, as they do not necessarily indicate restrictions of interactions, but rather show that

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there is a special relationship, which is likely to alter the incentives of the involved actors. Our model allows one to specify for example how one person treats a friend differently than a stranger. In addition, it is possible to study distributed graph clustering problems (such as, e.g., correlation clustering [21]) within this framework.

In many applications that can be modeled with our games there is a crucial lack of central coordination. Our main interest is thus how the set of players can quickly arrive at a stable set of decisions – a Nash equilibrium of the game – using distributed decision making policies. Our main result is that myopic sequential better response dynamics converge in polynomial time to a Nash equilibrium. This also holds for a payoff-relative concurrent protocol without central coordination. These results hold for local interaction games based on arbitrary symmetric 2×2 games and arbitrary graphs, which encompasses the vast majority of cases considered previously in related work. For the more general model of 2-type interaction games with symmetric 2×2 games and arbitrary graphs, we can also show polynomial time convergence of sequential dynamics. While sequential better response has a natural and intuitive appeal, our concurrent policy is carefully designed. It exhibits a number of favorable properties, such as respecting player incentives and relying only on local information. Designing such policies that yield provably rapid convergence is a major concern in wireless networks and distributed control systems (see, e.g. [16,31]), and our results contribute to this research agenda. As a byproduct, our dynamics yield efficient algorithms to compute a Nash equilibrium, which stands in sharp contrast to other game-theoretic models of restricted (graphical) interaction [12,27].

The comparison of convergence times for sequential and concurrent dynamics in local interaction games without dominant strategy reveals that the lack of central control can result in concurrent dynamics being slower than sequential ones. This, however, is a worst-case result, and we indicate that in coordination games concurrent dynamics resulting from our protocol can be significantly faster. This does not necessarily hold for anti-coordination games, and here a simple adjustment of our concurrent dynamics to a fixed choice μ for the migration probability can yield better results. However, the choice of this value is delicate, as resulting dynamics might abruptly drown in oscillation. It remains an interesting open problem to find improved analytical bounds for expected convergence times in specific classes of local interaction games.

The rest of the paper is structured as follows. We revisit related work in Section 1.1 and define the model in Section 1.2. Sequential dynamics are treated in Section 2, concurrent dynamics in Section 3. In Section 4 we compare sequential and concurrent convergence times in simple local interaction games. Finally, Section 5 concludes the paper. Missing proofs are given in the Appendix.

1.1 Related Work

This paper fits into a recent stream of works that study subclasses of local interaction games. For example, our model of local interaction games generalizes a game considered by Bramoullé [8], which concentrates on the subclass of symmetric 2×2 anti-coordination games on the edges and does not have any games on the non-edges. A special class of anti-coordination game derived from the MaxCut problem has been used in [15]. It was studied by Christodoulou et al. [11] in terms of convergence time to Nash equilibria and social welfare of states obtained after a polynomial number of best response steps.

Variants of local interaction games with coordination games are central in the study of threshold phenomena, cascading dynamics, and information diffusion in networks [30]. Closest to our focus is a recent paper by Montanari and Saberi [34] who consider local interaction games with 2×2 symmetric coordination games and a class of noisy best response dynamics called logit-response, heat bath, or Glauber dynamics. For potential games, in the long run, the time this process spends at a state scales proportional to the potential value and the noise level. For small noise levels the dynamics thus remain exclusively at global potential maximizers. For coordination games this is a state in which all players use the same strategy. The results of [34] are complementary to ours in the sense that they consider the hitting time of a *global* potential maximizer in a significantly more restrictive model. They show that convergence times increase from polynomial to exponential time when the graph becomes more well-connected. This contrasts our polynomial time bounds for all graphs and arbitrary symmetric 2×2 games when only convergence to *local* potential maximizers is required. In a related work, Kearns and Tan [29] design a voting protocol with polynomial time convergence in a similar 2-strategy coordination scenario. In contrast to our work they also require collective unity of choices.

While in our model the graph is fixed and specified in advance, there are several works on games with network formation. In particular, 2×2 anti-coordination games on endogenous graphs were studied in [9]. Much more work [6, 10, 13, 37] has been done on network formation and 2×2 coordination games. These games are classes of local interaction games with network creation, i.e., they allow only connected players to interact. There has been no focus on duration of dynamics, social welfare, and computation of Nash equilibria and optimal states. Instead, properties of the network structure and payoff properties in Nash equilibria were analyzed [4], or stochastically stable states were characterized [23, 25].

In the graphical model of evolutionary game theory introduced by Kearns and Suri [28] all players play a 2-player symmetric game with a randomly chosen neighbor. The authors characterize evolutionary equilibria in terms of the graph structure. However, they give no notion of dynamics that converge to equilibrium.

Our concurrent dynamics are closely related to recent work on protocols for concurrent strategy updating in potential games for distributed control in networks [31,32], some of which are inspired by evolutionary game theory [1,16,17]. In addition, there is a large body of related work on strategic learning [20,38], various forms of dynamics such as calibrated [18] or regret learning [5,24,39] or best response/ficticious play [2,11,15,19,33], and a variety of equilibrium concepts such as correlated Nash [3] or sink equilibria [14, 22].

1.2 Model and Notation

We begin by giving the formal definition of a 2-type interaction game.

Definition 1. *A* 2-type interaction game *is a graph $G = (V, E)$ together with two, possibly different, 2-player symmetric games Γ^c and Γ^d, where the set of strategies is the same in both games.*

Intuitively, on each edge $e \in E$ connected players play a 2-player symmetric game Γ^c. In addition, for each non-edge the pair of disconnected players play a possibly different symmetric 2×2 game, Γ^d. Each player plays the same strategy in all games he is participating. In this work, we restrict Γ^c and Γ^d to be 2×2 symmetric games with strategies 1 and 2, and payoffs for Γ^c and Γ^d are denoted as shown in Figure 1.

Next we introduce some notation that will allow us to define the utility function for each player in a 2-type interaction game. We will let $\Gamma_p = \{\Gamma^c, \Gamma^d\}$. We denote by $n = |V|$ the number of players, $m = |E|$

Γ^c	1	2
1	a,a	d,c
2	c,d	b,b

Γ^d	1	2
1	e,e	h,g
2	g,h	f,f

Figure 1: Payoffs in the Games.

the number of edges, $\deg(v)$ the degree of player v. Let $S = \{1, 2\}^n$ be the set of states of the game and $s = (s_v)_{v \in V} \in S$ a state, where $s_v \in \{1, 2\}$ is the strategy of player v. For a state s the set of players playing strategy 1 is denoted V_1, their number $n_1 = |V_1|$. For a player v the number $\deg_1(v)$ denotes the number of her neighbors playing 1, and n_1^{-v} the number of players except v that play strategy 1. V_2, n_2, $\deg_2(v)$, and n_2^{-v} are defined similarly for strategy 2. The size of the cut of a state s, which is the number of edges connecting players that play different strategies, is denoted by m_{12}. A player v has utility for strategy 1

$$\text{util}_v(1, s_{-v}) = [\mathtt{a} \cdot \deg_1(v) + \mathtt{d} \cdot \deg_2(v)] + [\mathtt{e} \cdot (n_1^{-v} - \deg_1(v)) + \mathtt{h} \cdot (n_2^{-v} - \deg_2(v)))]$$

while for strategy 2 he has utility

$$\text{util}_v(2, s_{-v}) = [\mathtt{c} \cdot \deg_1(v) + \mathtt{b} \cdot \deg_2(v)] + [\mathtt{g} \cdot (n_1^{-v} - \deg_1(v)) + \mathtt{f} \cdot (n_2^{-v} - \deg_2(v))].$$

Note that symmetric 2×2 games are known to be potential games [6, 37], and the potential is given as follows:

$$\Phi^c = \begin{pmatrix} \mathtt{a} - \mathtt{c} & 0 \\ 0 & \mathtt{b} - \mathtt{d} \end{pmatrix} \qquad \Phi^d = \begin{pmatrix} \mathtt{e} - \mathtt{g} & 0 \\ 0 & \mathtt{f} - \mathtt{h} \end{pmatrix}. \tag{1}$$

Here the potential for two players playing strategies i and j respectively, where $i, j \in \{1, 2\}$, is $\Phi^c(i, j)$ for Γ^c and $\Phi^d(i, j)$ for Γ^d. Note that each game has a potential function $\Phi(s)$ defined as sum of the corresponding potential values Φ^c and Φ^d of the 2-type interaction games.

2 Sequential Dynamics

In this section, our goal is to examine the duration of sequential iterative better response dynamics. We provide an analysis of the potential function, whichs yield polynomial convergence times in 2-type interaction games.

Theorem 1. *For every 2-type interaction game every sequence of better response moves from any initial state terminates in a pure Nash equilibrium after at most* $(n + 1)(m + 1)^2$ *steps.*

Proof. Our proof relies on a more insightful characterization for the potential function. We will simplify the games by subtracting $\mathtt{c}$ and $\mathtt{g}$ from every entry of the Γ^c and Γ^d, respectively. This does not alter payoff differences for the players and preserves the incentives. We can, in turn, derive yet another game equivalent to this one which has the doubly symmetric form described by Figure 2. We use $\mathtt{A} = \mathtt{a}$, $\mathtt{B} = \mathtt{b} - \mathtt{d}$, $\mathtt{E} = \mathtt{e}$, and $\mathtt{F} = \mathtt{f} - \mathtt{h}$. As shown in Chapters 1 and 2 of [36] the new game exhibits the same potential and Nash Equilibria as the original game. Note that this game is not equivalent in terms of social welfare, as we alter the total payoffs in some of the states.

Γ^c	1	2
1	A, A	0,0
2	0,0	B, B

Γ^d	1	2
1	E, E	0,0
2	0,0	F, F

Figure 2: Payoffs in games transformed to be doubly symmetric.

We analyze the underlying characteristic function more closely and denote by $\mathtt{S} = \mathtt{A} + \mathtt{B}$, $\mathtt{T} = \mathtt{E} + \mathtt{F}$, and $\Delta\mathtt{A} = \mathtt{A} - \mathtt{E}$. The potential function of Γ_p is

$$\begin{aligned} \Phi(s) &= \sum_{v \in V_1} \deg_1(v)\mathtt{A} + (n_1 - 1 - \deg_1(v))\mathtt{E} + \sum_{v \in V_2} \deg_2(v)\mathtt{B} + (n_2 - 1 - \deg_2(v))\mathtt{F} \\ &= n(n-1)\mathtt{F} + 2m(\mathtt{B} - \mathtt{F}) + \mathtt{T}n_1^2 - (2(n-1)\mathtt{F} + \mathtt{T})n_1 + (\mathtt{T} - \mathtt{S})m_{12} \\ &\quad + (\mathtt{T} - \mathtt{S} + 2\Delta\mathtt{A}) \sum_{v \in V_1} \deg(v) \end{aligned}$$

It is possible to drop the constant terms $n(n-1)\mathtt{F} + 2m(\mathtt{B} - \mathtt{F})$ from every state and derive a characteristic function $\Psi(s)$ given by

$$\Psi(s) = \mathtt{T}n_1^2 - (2(n-1)\mathtt{F} + \mathtt{T})n_1 + (\mathtt{T} - \mathtt{S})m_{12} + (\mathtt{T} - \mathtt{S} + 2\Delta\mathtt{A}) \sum_{v \in V_1} \deg(v). \tag{2}$$

This function depends - in addition to the payoffs - only on three parameters: the number n_1 of players playing strategy 1, their degrees $\sum_{v \in V_1} \deg(v)$ and the cut size m_{12}. $\Psi(s)$ becomes a potential for all 2×2

games by plugging in the payoffs of the games Γ^c and Γ^d into parameters $\mathtt{A}$, $\mathtt{B}$, $\mathtt{E}$ and $\mathtt{F}$ as described above. Then if we let $\mathtt{S} = \mathtt{a} + \mathtt{b} - \mathtt{d}$, $\mathtt{T} = \mathtt{e} + \mathtt{f} - \mathtt{h}$ and $\Delta\mathtt{A}_p = \mathtt{a} - \mathtt{e}$, we get a potential function

$$\Psi_p(s) = \mathtt{T}n_1^2 - ((n-1)(2\mathtt{f} - 2\mathtt{h}) + \mathtt{T})n_1 + (\mathtt{T} - \mathtt{S})m_{12} + (\mathtt{T} - \mathtt{S} + 2\Delta\mathtt{A}_p) \sum_{v \in V_1} \deg(v). \tag{3}$$

For the proof of the theorem observe that n_1 can range from 0 to n, which constitutes the factor $n+1$ in the guarantee. Note that m_{12} and $\sum_{v \in V_1} \deg(v)$ can take at most $m+1$ different values each. Hence, the total number of possible combinations for these parameters yields a total of $(n+1)(m+1)^2$ different values for Φ. As each better response iteration must strictly increase Φ in each step, every such sequence takes at most this number of iterations to reach a local optimum of Φ, from any starting state. This proves the theorem. □

The main technique in the previous proof is transforming any game to an equivalent doubly symmetric game with only four different payoff values. The main outcome of this is the function Ψ in Equation 2. By using the correct payoff values, it becomes Ψ_p, a potential function for our original game with an insightful representation.

The basis of the previous proof is a simple argument that can be applied somewhat more generally. Suppose every pair of players plays an exact potential game, each player can pick his strategy only once for all games, and the payoffs he receives are summed up. Then the whole game has an exact potential function. Consider a local interaction game in which each pair of players plays a $k \times k$ potential game with constant k. We can classify edges into $O(k^2)$ classes depending on the current state of the game on the edge. This yields only a polynomial number of different combinations and potential values. The same holds if we generalize 2-type interaction games to a constant number of different $k \times k$ potential games with constant k. On the other hand, if we allow on each edge a different game, then even with $k = 2$ we can encode local search in instances of weighted MaxCut, and therefore worst-case convergence time becomes necessarily exponential. Similarly, in a local interaction game with $k \times k$ games and $k \leq n$, it is possible to encode an instance of weighted MaxCut simultaneously into payoff matrix and graph structure for a subgraph of $k/2$ nodes. Thus, for $k = \Omega(n)$ strategies this yields games, in which convergence time is necessarily exponential. A deeper characterization along these lines is left for future work.

3 Concurrent Dynamics

In this section we consider round-based concurrent dynamics, in which in each round all players simultaneously update their strategy choices. A simple approach, which is considered frequently in the area of information diffusion in networks [30], is to allow all players simultaneously play their best responses to the current state of the game. This approach converges rapidly if all players have dominant strategies. In fact, we would reach the dominant strategy equilibrium after the first round, which speeds up the convergence time by a factor of n over the sequential process considered previously. One might think that concurrent dynamics should always yield a speed-up of $\Theta(n)$ due to the possibility of simultaneous updates. However, due to the absence of global coordination, these dynamics can easily get stuck in oscillations. The main design challenge proves to be to avoid oscillation and to obtain reasonable convergence times. In order to do this we follow the idea of [16] and design a policy in order to increase the potential function in expectation in each round. The challenge here is to enlarge migration probabilities to converge quickly, yet to guarantee potential increase in expectation.

To guarantee convergence we introduce the notion of inertia. Suppose each player independently at random migrates to a better response with a probability less than 1. This allows for the construction of a Markov chain on the states, where migration probabilities of the players yield transition probabilities between states. Note that, due to inertia, with a possibly tiny probability the concurrent process can resemble any sequential better response dynamics. Thus, the only absorbing states of the Markov chain are the pure Nash equilibria, to which the process must converge with probability 1 in the limit (see, e.g., [32]). The bounds on the convergence time that can be derived from this argument, however, are usually extremely large.

Subsequently, we analyze a protocol with migration probabilities proportional to the relative payoff increase. For technical reasons, we here assume that all payoffs are non-negative integer numbers, i.e. $\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{e}, \mathtt{f}, \mathtt{g}, \mathtt{h} \in \mathbb{N}$. Afterwards, we consider several preprocessing steps to adjust the payoff values such that the incentives of players are preserved and convergence is obtained in expected polynomial time.

Algorithm 1 Relative Migration Protocol (RMP), repeatedly executed by all players in parallel.

For player v let $x \leftarrow s_v$ and $y \leftarrow 3 - x$.
if $\text{util}_v(y, s_{-v}) > \text{util}_v(x, s_{-v})$ **then**
  with probability

$$\mu_{xy} = \frac{1}{\lambda} \cdot \frac{\text{util}_v(y, s_{-v}) - \text{util}_v(x, s_{-v})}{\text{util}_v(y, s_{-v})}$$

  migrate from strategy x to y.
end if

In a state s a player v considers changing from strategy $x \in \{1, 2\}$ to strategy $y = 3 - x$ if $\text{util}_v(y, s_{-v}) - \text{util}_v(x, s_{-v}) > 0$. If this is the case, she migrates with migration probability that depends on her relative payoff increase (see the Relative Migration Protocol (RMP), Algorithm 1). If every player updates his strategy choices using the RMP, a new state s' evolves. We define a vector $\Delta s = (s'(v) - s(v))_{v \in V}$.

Lemma 1. *If $\mathtt{c} = \mathtt{d}$, $\mathtt{g} = \mathtt{h}$, and λ chosen sufficiently large, then as long as the 2-type interaction game is not in a Nash equilibrium, it holds that $\mathbb{E}[\Phi(s + \Delta s)] > \Phi(s)$.*

Say player v could improve his utility by switching to a new strategy. He decides to switch with a probability based on the action profile of his neighbors. At the same time as v changes strategy, his neighbors might do so as well. Thus this proof works by bounding the error in how much v expects to gain before switching versus how much v actually gains after switching.

Proof. For a state s and a vector Δs consider a player v. Let $y = s(v)$ denote v's current strategy and let $x = s(v) + \Delta s(v)$ denote v's strategy after migration. The change in v's utility after migration, assuming no other players change their strategy is denoted $\Delta\text{util}_v(s_{-v}) = \text{util}_v(y, s_{-v}) - \text{util}_v(x, s_{-v})$. Let the *virtual potential gain* be defined as $VP(s, \Delta s) = \sum_{v \in V} \Delta\text{util}_v(s_{-v})$. The virtual potential simply sums all the presumed payoff increases of all players that chose to migrate. The real potential gain $\Phi(s + \Delta s) - \Phi(s)$ can be different if more than a single player moves. In this case the simultaneous migration of players u and v creates an error $F^{u,v}(s, \Delta s)$. Thus,

$$\Phi(s + \Delta s) - \Phi(s) = VP(s, \Delta s) - \sum_{u,v \in V} F^{u,v}(s, \Delta s). \tag{4}$$

In order to show that $\mathbb{E}[\Phi(s+\Delta s)] - \Phi(s) > 0$, and conclude the proof of Lemma 1, we will relate expected virtual potential gain and expected error, which are the two terms on the right hand side of Equation 4. We denote by $\lambda^* = \gamma \cdot \frac{1}{2} \cdot \left(1 + \max\left\{\frac{\mathtt{a}}{\mathtt{b}}, \frac{\mathtt{b}}{\mathtt{a}}, \frac{\mathtt{e}}{\mathtt{f}}, \frac{\mathtt{f}}{\mathtt{e}}\right\}\right)$, where $\gamma > 1$ is a constant.

Lemma 2. *For $\lambda > \lambda^*$ it holds that $\mathbb{E}[\Phi(s + \Delta s)] - \Phi(s) \geq \frac{\gamma - 1}{\gamma} \cdot \mathbb{E}[VP(s, \Delta s)]$.*

Proof. We will show that the error terms $\sum_{u,v \in V} \mathbb{E}[F^{u,v}(s, \Delta s)]$ are at most a constant fraction of $\mathbb{E}[VP(s, \Delta s)]$, and the lemma will follow by taking the expection of Eqn. (4). We will account the expected virtual potential gain partially to each pair of nodes $u, v \in V$, and thereby relate it to the expected error of the potential gain between u and v. For simplicity we drop the indices s, Δs and s_{-v}. Note that

$$\begin{aligned}
\mathbb{E}[VP] = \sum_{v \in V} \mu_{xy}^v \cdot \Delta\text{util}_v \quad &= \quad \frac{1}{\lambda} \sum_{v \in V} \frac{(\Delta\text{util}_v)^2}{\text{util}_v(y)} \\
&= \quad \frac{1}{\lambda} \sum_{v \in V} \Delta\text{util}_v \cdot R_v \quad ,
\end{aligned}$$

where μ_{xy} is defined in Algorithm 1 and we use $R_v = \Delta\text{util}_v/\text{util}_v(y)$. We split this expected virtual potential gain into parts denoted $VP^{u,v}$, which are accounted towards the pair (u,v) of players, for every $u \neq v$, $u,v \in V$. For a player v we account a fraction of his gain depending on the payoff that the game with player u contributes to $\text{util}_v(y)$.

The following analysis is done for a player v with $s(v) = x = 1$ that migrates to strategy $y = 2$ and pairs of neighboring players. The arguments can be repeated similarly for a switch from 2 to 1 and/or disconnected players. We first consider a neighbor u with $s(u) = 2$. For player v we account a fraction of

$$\frac{\texttt{b}}{\text{util}_v(2)} \cdot \mu_{12}^v \cdot \Delta\text{util}_v = \texttt{b} \cdot \frac{1}{\lambda} \cdot \left(\frac{\Delta\text{util}_v}{\text{util}_v(2)}\right)^2 = \frac{\texttt{b}}{\lambda} \cdot R_v^2$$

of the expected virtual potential gain to the edge (u,v). Similarly, u has an incentive to change from strategy 2 to 1, and we account a fraction of

$$\frac{\texttt{a}}{\text{util}_u(1)} \cdot \mu_{21}^u \cdot \Delta\text{util}_u = \texttt{a} \cdot \frac{1}{\lambda} \cdot \left(\frac{\Delta\text{util}_u}{\text{util}_u(1)}\right)^2 = \frac{\texttt{a}}{\lambda} \cdot R_u^2$$

of the expected virtual potential gain to (u,v). Thus, we have

$$\mathbb{E}\left[VP^{u,v}\right] = \frac{1}{\lambda} \cdot (\texttt{a}R_u^2 + \texttt{b}R_v^2).$$

The expected error is calculated as follows. Player v presumes a change in payoff of $\texttt{b} - \texttt{c}$, player u presumes $\texttt{a} - \texttt{d} = \texttt{a} - \texttt{c}$. However, if both players migrate the combined change in potential is $(\texttt{a} - \texttt{c}) + (\texttt{c} - \texttt{a}) = 0$. Thus, the error is $\texttt{a} + \texttt{b} - 2\texttt{c}$, and the expected error is

$$\mathbb{E}\left[F^{u,v}\right] = \mu_{21}^u \mu_{12}^v \cdot (\texttt{a} + \texttt{b} - 2\texttt{c}) \leq \frac{\texttt{a} + \texttt{b}}{\lambda^2} \cdot R_u \cdot R_v \ .$$

Thus, the expected potential gain for the pair (u,v) is at least

$$\mathbb{E}\left[VP^{u,v}\right] - \mathbb{E}\left[F^{u,v}\right] \geq \frac{1}{\lambda}\left(\texttt{b}R_v^2 + \texttt{a}R_u^2 - \frac{(\texttt{a} + \texttt{b})}{\lambda} R_u R_v\right) \ . \tag{5}$$

This expression is strictly positive if we ensure that $(\texttt{a}+\texttt{b})/\lambda < 2\min\{\texttt{a},\texttt{b}\}$, which yields $\lambda > \frac{1}{2}\cdot\left(1 + \max\left\{\frac{\texttt{a}}{\texttt{b}}, \frac{\texttt{b}}{\texttt{a}}\right\}\right)$. By our choice of $\lambda > \lambda^*$ this is guaranteed and yields

$$\mathbb{E}\left[F^{u,v}\right] \leq \frac{1}{\gamma} \cdot \mathbb{E}\left[VP^{u,v}\right] \ .$$

This proves the case for a neighbor u with $s(u) = 2$.

The case for a neighbor u with $s(u) = 1$ follows similarly. For player v we account a fraction of

$$\frac{\texttt{c}}{\text{util}_v(2)} \cdot \mu_{12}^v \cdot \Delta\text{util}_v = \frac{\texttt{c}}{\lambda} \cdot R_v^2$$

of his expected virtual potential gain to the edge (u,v). Similarly, u has an incentive to change from strategy 1 to 2, and we account a fraction of

$$\frac{\texttt{c}}{\text{util}_u(2)} \cdot \mu_{12}^u \cdot \Delta\text{util}_u = \frac{\texttt{c}}{\lambda} \cdot R_u^2$$

of the expected virtual potential gain to (u,v). Thus, we have

$$\mathbb{E}\left[VP^{u,v}\right] = \frac{\texttt{c}}{\lambda} \cdot (R_u^2 + R_v^2)$$

The expected error on edge (u, v) is calculated as follows. Each player presumes a change in payoff of $\mathtt{c} - \mathtt{a}$, however, if both players migrate, the change in potential is $\mathtt{c} - \mathtt{a} + \mathtt{b} - \mathtt{c} = \mathtt{b} - \mathtt{a}$. Thus, the error is $2(\mathtt{c} - \mathtt{a}) + (\mathtt{a} - \mathtt{b}) = 2\mathtt{c} - \mathtt{a} - \mathtt{b}$, and the expected error is

$$\mathbb{E}\left[F^{u,v}\right] = \mu_{12}^{u}\mu_{12}^{v} \cdot (2\mathtt{c} - \mathtt{a} - \mathtt{b}) \leq \frac{2\mathtt{c}}{\lambda^2} \cdot R_u \cdot R_v \ .$$

Due to the consistent factor $\mathtt{c}$ in this case we can actually argue with $\mathtt{b}, \mathtt{c} \geq 0$ and simple calculus that for any constant $\lambda > 1$ we have

$$\mathbb{E}\left[F^{u,v}\right] \leq \frac{1}{\lambda} \cdot \mathbb{E}\left[VP^{u,v}\right] \leq \frac{1}{\gamma} \cdot \mathbb{E}\left[VP^{u,v}\right] \ .$$

The same argument can be repeated for all pairs of players and all possible strategy constellations. Finally, we see that

$$\sum_{u,v \in V} \mathbb{E}\left[F^{u,v}\right] \leq \frac{1}{\gamma} \cdot \mathbb{E}\left[VP\right]$$

This combined with Equation 4 proves Lemma 2. □

Note that, as long as at least one payoff value of $\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}$ is strictly positive, we make a strictly positive increase in the potential function whenever a player moves. This proves Lemma 1. □

In the following we will adjust local interaction games such that we preserve the incentives of players and the dynamics resulting from the RMP converge to a Nash equilibrium in expected polynomial time. We first turn the games into doubly symmetric games of the form in Figure 2. We then use the fact that for any such local interaction game we can find an equivalent game, in which $\mathtt{A}, \mathtt{B} \in [-2n^2, 2n^2]$, which we prove below. Finally, adjusting doubly symmetric games of Fig. 2 to ensure that all payoffs are positive is straightforward by adding $2\max\{|\mathtt{A}|, |\mathtt{B}|\}$ to every payoff value. Note that this adjustment at most triples the maximum absolute value of all payoffs. When we run the RMP with payoffs adjusted in this way, we can guarantee polynomial convergence time. We refer to this as the *perturbed RMP*.

Theorem 2. *For local interaction games the dynamics resulting from the perturbed RMP converge to a Nash equilibrium in expected polynomial time.*

Proof. We first observe that we can always replace payoff values by numbers in $O(n^2)$ that yield the same player incentives. Then we show that this adjustment results in polynomial convergence time.

Let us consider the game Γ_p of the form in Fig. 2. Note that if $\mathtt{A} \geq 0 \geq \mathtt{B}$ or vice versa, the game has a weakly dominant strategy and we get an equivalent game with $\mathtt{A}, \mathtt{B} \in \{1, 0, -1\}$. For $\mathtt{A}, \mathtt{B} > 0$ (the negative case is similar) consider any player $v \in V$. The number of neighbors of v playing strategy 1 or 2 define the payoffs and the preferences of v. They are given by $\deg_1(v)\mathtt{A} \geq \deg_2(v)\mathtt{B}$ or $\deg_1(v)\mathtt{A} \leq \deg_2(v)\mathtt{B}$, with $\deg_1(v) + \deg_2(v) \leq n - 1$, $\deg_1(v), \deg_2(v) \geq 0$. These inequalities can be transformed into upper and lower bounds for $\mathtt{A}/\mathtt{B}$. The tightest upper and lower bounds for this expression reveal that $\mathtt{A}/\mathtt{B}$ can be discretized into a polynomially bounded rational number. In particular, by choosing values for $\mathtt{A}, \mathtt{B} \in \{1, \ldots, 2n^2\}$ it is always possible to satisfy the tightest (and thereby all) upper and lower bounds. This preserves all possible preferences in the game and reduces the payoffs to values in $O(n^2)$.

When payoffs are bounded by $O(n^2)$, then $\lambda^* \in O(n^2)$. Thus, the expected virtual potential gain in each round of the concurrent dynamics is in $\Omega(1/n^4)$, and so is the expected potential increase. Examining the potential for local interaction games with payoffs in $O(n^2)$ reveals that the maximum value of the potential is bounded by $O(n^4)$. The expected time to reach a state of maximum potential is thus bounded by $O(n^8)$. □

We strongly believe that a similar reduction to polynomial payoff values is also possible in the case of 2-type interaction games. The technical details are quite tedious and an analysis of this case is omitted. It is, however, straightforward to argue that if an 2-type interaction game has payoffs polynomial in n, i.e., in $O(n^k)$ for some constant k, then $\lambda^* \in O(n^k)$ and the perturbed RMP yields an expected potential increase in each iteration that is in $\Omega(n^{-(2k+1)})$. As in this case the maximum potential value is in $O(n^{k+2})$, we have the following corollary.

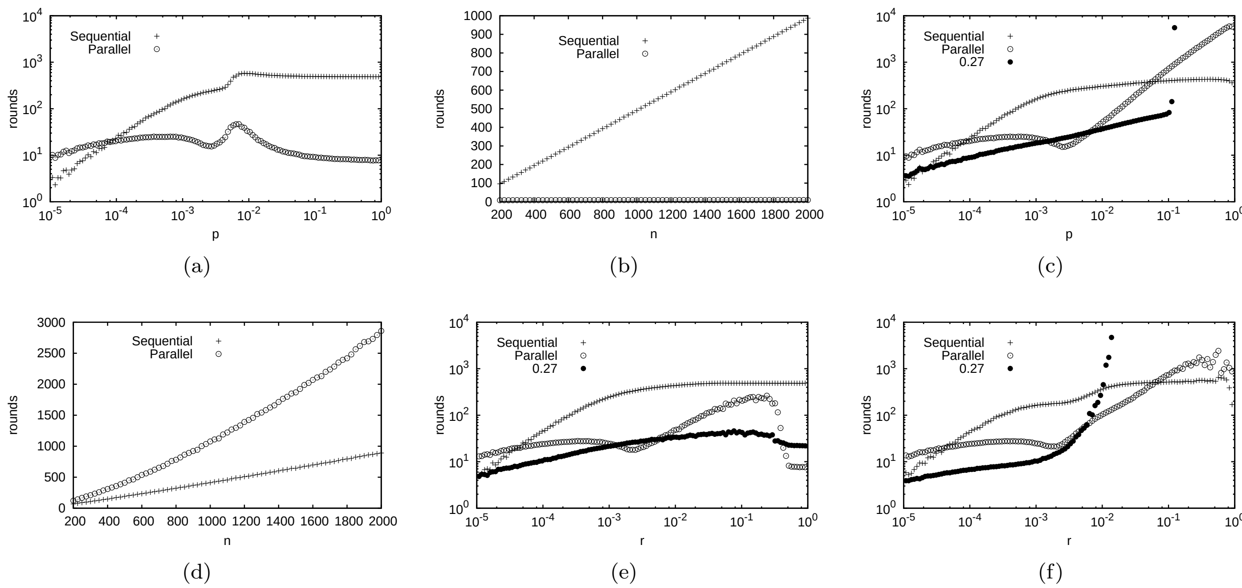

Figure 3: Running times of sequential and concurrent dynamics. (a) Coordination games on $G_{n,p}$ with $n = 1000$ and varying p. (b) Coordination games on $G_{n,p}$ with $p = \log^{-1}(n)$ and varying n. (c) Anti-coordination games on $G_{n,p}$ with $n = 1000$ and varying p. (d) Anti-coordination games on $G_{n,p}$ with $p = \log^{-1}(n)$ and varying n. (e) Coordination games on random unit disk graphs with $n = 1000$ and varying radius r. (f) Anti-coordination games on random unit disk graphs with $n = 1000$ and varying radius r.

Corollary 1. *For 2-type interaction games with payoffs bounded by $O(n^k)$ with a constant k the dynamics resulting from the perturbed RMP converge to a Nash equilibrium in expected polynomial time.*

4 Comparison of Convergence Times

The bound on convergence times presented in previous sections hold in general for any 2-type interaction game. However, there are significant differences between different types of games. We will exhibit these differences experimentally using the simpler local interaction games. In dominant strategy games concurrent dynamics have an obvious advantage, because there is no error when allowing players to migrate. In particular, by appropriately adjusting payoffs to 0 and 1 we can ensure that in the RMP every player migrates with probability 1 to the dominant strategy. The details are left as exercise to the reader.

If there is no (weakly) dominant strategy, the game Γ_p is either a coordination game with $\mathtt{A}, \mathtt{B} > 0$, or an anti-coordination game with $\mathtt{A}, \mathtt{B} < 0$. For simplicity we restrict to *elementary* games, in which $\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d} \in \{0, 1\}$. For such games it is possible to show a time bound of $O(n^2)$ for sequential dynamics, and of $O(n^3)$ for concurrent dynamics resulting from the RMP, which will will call *RMP dynamics*.

4.1 Coordination Games

First we consider elementary coordination games with $\mathtt{a} = \mathtt{b} = 1$ and $\mathtt{c} = \mathtt{d} = 0$. The worst-case upper bound for the convergence time of RMP dynamics is a factor of $\Theta(n)$ larger. It is left as an exercise to design a game matching this difference, i.e. a game in which the RMP dynamics are a factor of $\Omega(n)$ slower than any sequential better response dynamics. In fact, there is a game in which *every* concurrent dynamics are at least as slow as *any* sequential dynamics (see Proposition 1 in Appendix 7). We contrast these worst-case results with the average-case behavior on random graphs generated according to the $G_{n,p}$ model, as is done in the work of [26, 28]. We will observe similar behavior also on random unit disk graphs below. It turns out that in these games aggressive concurrent dynamics can make very rapid progress in initial stages. For a proof see the Appendix.

Theorem 3. *Let $0 \leq c < 1$ be a constant and $\frac{1}{n^c} \leq p \leq \frac{1}{2}$, and let G be generated via $G_{n,p}$. Consider a state in the elementary coordination game with at least $(1/2+\delta)n$ nodes playing strategy 1 and at most $(1/2-\delta)$ nodes playing strategy 2, where $1/2 \geq \delta \geq 0$ is a constant. After 1 round of concurrent best response dynamics all but $o(n)$ nodes will be playing strategy 1.*

If the dynamics are sequential instead of concurrent, one can show by a similar argument to the above that after n rounds all but $o(n)$ nodes will be playing strategy 1.

Next, we show a number of experimental results in Fig. 3. For each value of n and p we generated 10 random graphs, and on each random graph we chose 25 starting states uniformly at random. From each starting state we initiated 25 runs of RMP dynamics. For the sequential dynamics we deterministically chose in each round one player that yields the largest payoff gain. The constant λ was set to $\lambda = 1.1$ throughout. Fig. 3(a) shows the average number of rounds for $n = 1000$ and p increasing exponentially between 10^{-5} and 1. When the large component forms (around $p = 0.005$) the sequential times are close to $n/2$, while the RMP dynamics converge rapidly in a constant number of runs.

Although Theorem 3 does not directly bound the convergence time to Nash equilibria, it provides the main intuition for the explanation of the results. After random initialization there are close to $n/2$ players playing each strategy. Afterwards, due to similar neighborhoods and coordination structure of the game, nearly all players accumulate on one strategy. Although this does not happen in one step, it still occurs quite rapidly, as each player migrating to a predominant strategy increases the probability for others to follow. Thus, in essence the behavior of the RMP dynamics is characterized by the insights from Theorem 3.

The intuition follows similarly for the sequential case, see Fig. 3(b). It depicts running times on graphs with increasing n and $p = \log^{-1}(n)$. Observe that RMP dynamics yield rapid convergence times that increase only very slightly. Sequential dynamics need roughly $\Theta(n)$ rounds until a Nash equilibrium is reached.

4.2 Anti-Coordination Games

The elementary anti-coordination game is the MaxCut game with $\mathtt{a} = \mathtt{b} = 0$ and $\mathtt{c} = \mathtt{d} = 1$. For this game the worst-case results are similar to the coordination case. More specificaly, RMP dynamics are be a factor of $\Omega(n)$ slower than any sequential better response dynamics, and the game reveals that every concurrent dynamics are at least as slow as any sequential dynamics (see Proposition 2 in Appendix 7). We complement this lower bound with experimental results in Fig. 3. Fig. 3(c) and 3(d) are generated using the same parameters as for Fig. 3(a) and 3(b), respectively. While for small p the behavior of both dynamics is similar to the coordination case, it changes when $p \geq \frac{1}{n^c}$ for $c < 1$ which corresponds to roughly $p \geq 10^{-2}$ in Fig. 3(c). Observe the linear increase in running time with growing p for the RMP dynamics, which for large p leads even to worse convergence times than for sequential dynamics. A linear dependence on p is also supported by Fig. 3(d), as here $p = \log^{-1}(n)$, and the time growth for the RMP dynamics is proportional to $n \log n$. In fact, the linear dependence is a result from the RMP dynamics being too passive. Unlike in the coordination case, players do not accumulate on one strategy choice. In most iterations there is no significant majority playing one strategy. Payoff differences remain small, so with degrees growing linear in p, migration probabilities μ^v drop to a level proportional to $1/p$. The expected time until a player migrates then grows linearly in p. This effect is present until p is very close to 1, in which case the convergence times of sequential dynamics drop to 0, as uniformly random initialization yields an almost stable profile. Furthermore, for almost complete graphs, the RMP dynamics yield a sequential process with high probability. This is because, in very dense graphs almost all players have the same neighborhood and experience the same changes in payoff. The migration probabilities in the RMP dynamics of roughly $1/n$ are balanced by the $\Theta(n)$ players that are willing to migrate in each round, so there is a roughly constant number of player migrating in each round.

Large running times are due to the payoff-relative update rule of the RMP. With different choices it is possible to achieve much more rapid convergence. Fig. 3(c) also depicts the convergence times of concurrent dynamics on graphs with $n = 1000$ and varying p where all migration probabilities μ^v are chosen as a fixed value $\mu = 0.27$, other values yield similar results. The increased migration significantly decreases the expected running times below the sequential times. At some point, however, the dynamics rather abruptly hit an "oscillation barrier" and convergence times start growing exponentially. Characterizing this barrier and providing further analytical insights on suitable choices of migration probabilities in concurrent dynamics remains a fascinating open problem.

Finally, we note that the key observations hold similarly for the case of random unit-disk graphs, which are a popular model for interference in distributed networks. We generated graphs by placing n points uniformly at random in the unit square. An edge was created between two points if the distance under the maximum norm was at most r. For each graph we chose 25 starting states uniformly at random, and from each state we initiated 25 runs of the dynamics. We provide average running times in Figure 3(e) and 3(f).

5 Conclusion

We have studied distributed decision making in a fundamental class of network interaction games with various applications in distributed systems and social networking. Our results concern the convergence time of sequential and concurrent better response dynamics. The analysis reveals polynomial convergence times for sequential dynamics in both local interaction games and 2-type interaction games. For concurrent dynamics resulting from the RMP there is polynomial convergence time in local interaction games, and in 2-type interaction games with polynomially bounded payoffs. In these games a local potential maximizer – i.e. a pure Nash equilibrium – can be obtained efficiently using distributed protocols, and thus efficient distributed decision making is possible. This stands in contrast to noisy better response dynamics and global potential maximizers, which are NP-hard to compute in anti-coordination games. Even for coordination games, in which computation is trivial, noisy better response dynamics can take exponential time [34].

While our results establish a general upper bound, the actual convergence times differ significantly based on the type of interaction and the underlying network. Using experiments we have shed some light on the influence of incentives and the degree of connectedness. More work is needed to obtain analytical characterizations for specific games and graph classes of interest.

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6 Proof of Theorem 3

For convenience let us define the S-degree of a vertex as follows.

Definition 2. *For a graph $G = (V, E)$ and $S \subseteq V$, the S-degree of a vertex $v \in V$ is* $\deg(v, S) = |\{u \in V \mid (u, v) \in E, u \in S\}|$.

The theorem follows by showing that a large number of vertices have similar neighborhoods. This idea is formalized in the following lemma and is similar to an argument given in [28].

Lemma 3. *Let $0 \leq c < 1$ be a constant and $\frac{1}{n^c} \leq p \leq \frac{1}{2}$. Let $S \subseteq V$ be such that $|S| \geq \tau n$ for some constant $\tau > 0$. For any constant $1/6 \geq \epsilon > 0$ the number of nodes that have S-degree outside the range $(1 \pm \epsilon)p\tau n$ is at most $o(n)$.*

Proof. The main vehicle for proving this lemma is Theorem 2.14 of Bollobas [7]. First we check that the necessary assumptions of that theorem are met: $\frac{6\log(n)}{\epsilon^2 p} \leq \frac{6\log(n)n^c}{\epsilon^2} \leq \tau n \leq |S|$. The inequalities hold because $\frac{1}{n^c} \leq p$ and $c < 1$. Thus for the set of nodes with a significantly unexpected number of neighbors $Z_S = \{z \in V \setminus S : ||\Gamma(z) \cap S| - p\tau n| \geq \epsilon p\tau n\}$ the theorem gives us with $\frac{1}{n^c} \leq p$ and $c < 1$ that

$$|Z_S| \leq \frac{12\log(n)}{\epsilon^2 p} \leq \frac{12\log(n)n^c}{\epsilon^2} = o(n)$$

□

We now apply the lemma to show our theorem. Since there are at least $(1/2 + \delta)n$ nodes playing 1, by Lemma 3 the number of nodes playing 2 with V_1-degree outside the range $(1/2 + \delta)(1 \pm \epsilon)pn$ is at most $o(n)$. Similarly, since there are at most $(1/2 - \delta)n$ nodes playing 2, by Lemma 3 the number of nodes with V_2-degree outside the range $(1/2 - \delta)(1 \pm \epsilon)pn$ is at most $o(n)$. We can choose ϵ small enough such that $(1/2 + \delta)(1 - \epsilon)pn > (1/2 - \delta)(1 + \epsilon)pn$. Thus, all but $o(n)$ of the nodes playing strategy 2 have more neighbors playing strategy 1 than strategy 2, and the best response for all but $o(n)$ of the nodes playing 2 is to switch to 1. By an analogous argument we can show that all but $o(n)$ of the nodes playing strategy 1 have more neighbors that play strategy 1 than 2. Thus all but $o(n)$ of the nodes playing 1 will have the best response of continuing to play 1. □

7 Worst-Case Running Times of Concurrent Dynamics

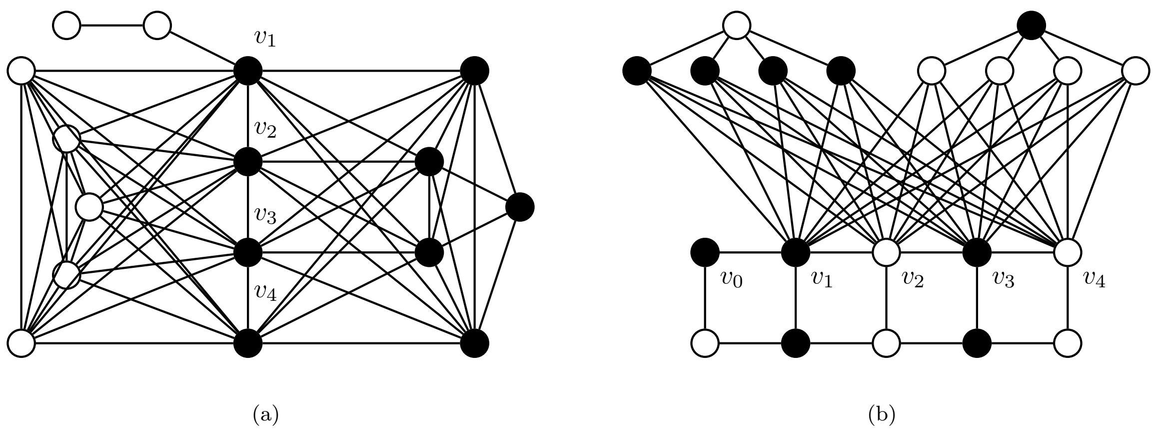

Figure 4: Elementary (a) coordination and (b) anti-coordination games establishing the lower bound with $k = 4$. Filled vertices play strategy 1, empty vertices play strategy 2. Starting with v players along the (a) middle and (b) upper line will switch to the opposite strategy one at a time.

Proposition 1. *For every $k \geq 1$ there is an elementary coordination game with $n = 3k+4$ players, in which every sequential better-response dynamics take exactly k steps and every concurrent dynamics take at least k steps. In particular, the RMP dynamics converge in $\Omega(kn)$ steps in expectation.*

Proof. Consider a game of the class depicted in Fig. 4(a). In this game we have a line of vertices $v_1, \ldots, v_k$, and all players currently play strategy 1. In addition, there are two cliques of size $k + 1$, in one clique all players play 1, whereas in the other clique all players play 2. Finally, there are two additional vertices that play strategy 2. Each vertex on the line is connected to all vertices of the clique that plays 2 and to $k - 1$ vertices of the clique that plays 1. Starting with v_1, in each round i the only player that wants to switch is player v_i from 1 to 2. Obviously, even if all players are given the possibility to jump, only this one player will migrate. For the RMP dynamics we observe that the relative improvement in payoff for the migrating player is in $\Theta(1/n)$ in each round, so it takes an expected number of $\Theta(n)$ rounds until one player migrates. □

This game reveals a fundamental dilemma for concurrent dynamics. On the one hand, it is possible to match the speed of sequential dynamics only if we let each player migrate with a large probability. On the other hand, frequent migration can yield long-lasting oscillations in other games. For the task of designing protocols with guaranteed rapid convergence, this problem is certainly critical. However, the bad properties are mainly due to the adversarial construction with an inherent partition into two parts that are intrinsically stable attached to opposite strategies.

Proposition 2. *For every $k \geq 1$ there is an elementary anti-coordination game with $n = 4k + 4$ players, in which every sequential better-response dynamics take exactly k steps and every concurrent dynamics take at least k steps. In particular, the RMP dynamics converge in $\Omega(kn)$ steps in expectation.*

Proof. The class of games we consider is depicted in Fig. 4(b). There are two *pure sets* of k vertices each, one set playing strategy 1, one set playing strategy 2. Each set of vertices is connected to one stabilizing vertex of the opposite strategy. In addition, there are two lines of length $k + 1$, which are connected into a grid. Players on the second line play strategies alternatingly, starting with strategy 2. On the first line $v_0, \ldots, v_k$ the assignment is the same except for the leftmost vertex v_0, which is assigned to strategy 1. In addition, all vertices v_i with $i \geq 1$ are connected to all vertices from both pure sets. Starting in this configuration, player v_1 is the only vertex that wants to switch. In the following, in round i only player v_i will switch to the opposite strategy. Again, even if all players are given the possibility to jump, only one player will migrate. For the RMP dynamics we observe that the relative improvement in payoff for the migrating player is in $\Theta(1/n)$ in each round, so it takes an expected number of $\Theta(n)$ rounds until one player migrates. □