A Network Formation Game for Bipartite Exchange Economies

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Abstract

We introduce a natural new network formation game in which buyers and sellers may purchase edges representing trading opportunities between themselves, and then accrue wealth in the resulting exchange economy. Our main result is an exact characterization of the set of bipartite graphs G that are Nash equilibria for this game. This characterization provides sharp limits on the amount and structure of wealth variation that can occur, as well as on the allowable equilibrium exchange rates.

1 Introduction

Recently there has been interest in both the computer science and economics communities in network formation games. Broadly speaking, in these multiplayer games, individuals may choose to share the cost of building a network by purchasing edges incident on themselves. Each player's overall utility consists of two, usually competing, components — on the one hand, the edge costs incurred by the player, and on the other, some measure of "benefit" accrued to the player by their participation or position in the network.

For instance, in one well-studied model [9, 1], individuals wish to minimize their edge purchases plus the sum of their (shortest-path) distances to all other players. Clearly there is a trade-off between these two components. Within such models there have been studies of the structural properties of those networks that are (pure strategy) Nash equilibria of the game, Price of Anarchy bounds, and other analyses. (See Related Work.)

As in the example above, in much of the prior research the benefit to a player for participating in the network measures some notion of their *centrality* or *connectivity* — shortest-path distances to other players, number of other players in the same component, and so on. In this paper we introduce and analyze a natural alternative — namely, we view the network formed by the players as defining *trading opportunities*, and measure the network benefit to a player by the wealth they accrue from those trading opportunities.

Our point of departure is a recently introduced networked version of classical exchange economies [14], and more specifically its specialization to bipartite buyer-seller networks [15]. In the latter, there is an exogenously specified bipartite network between n buyers (who each have an endowment of 1 divisible unit of an abstract commodity called cash) and n sellers (who each have an endowment of 1 divisible unit of an abstract commodity called wheat). Buyers have utility only for wheat and sellers only for cash, thus ensuring mutual interest in trade. The bipartite network is viewed as specifying all and only those pairs of buyers and sellers who may trade. Earlier work [14] established the existence of (market-clearing) equilibria in which prices and wealths may vary across the network due to topological asymmetries, paving the way for the later study [15] in which the network is generated according to standard stochastic (non-strategic) network generation models. There it was established (for example) that Erdos-Renyi networks exhibit essentially no price or wealth variation, while those generated according to preferential attachment have unbounded wealth variation (growing as a root of the population size).

In this paper we start with the same bipartite buyer-seller model, but now *endogenize* the creation of the network to arrive at a natural network formation game. More precisely, we assume that any buyer (respectively, seller) is free to purchase an edge to any seller (respectively, buyer) at a cost of α. The selection of which edges to purchase by all parties specifies an undirected bipartite network G, and in this network each party i achieves some exchange equilibrium wealth $\omega(G, i)$. Our network formation game is then defined by specifying the *overall* utility to i as

$$u_i = -\alpha \times e(G, i) + \omega(G, i)$$

where $e(G, i)$ is the number of edges in G purchased by i. We view the u_i as defining a one-shot, simultaneous move game over the $2n$ players, in which each player's action is a selection of which edges to purchase; see Section 2 for a formal definition and discussion of the game.

The network formation game given by the u_i is similar in broad spirit to previous network formation games, but quite different in its details. As with previous models, each player is balancing an outlay of wealth for edge creation (trading opportunities) against some resulting participatory benefit in G; but now the participatory benefit is measured in terms of wealth gained from trade rather than connectivity or shortest paths.

Our main results provide a precise structural characterization of all the networks G that are Nash equilibria of the game defined by the payoffs u_i above. More precisely, we establish exact conditions on the amount of exchange equilibrium wealth variation that can occur for any given values of α and n, and show that this in turn sharply limits the connectivity structure of any Nash equilibrium network G. We then show that these limits are tight by demonstrating specific Nash equilibrium networks G that saturate them, thus yielding a comprehensive catalog of all Nash equilibria. The resulting characterization also places sharp limits on the possible exchange rates or prices that are possible. For example, while with an exogenously specified graph, any rational exchange rate can be achieved, only very specific exchange rates can be achieved in a graph that is a Nash equilibrium of the formation game — an exchange rate of $2/5$, for instance, is impossible.

To our knowledge this is the first network formation game of comparable complexity for which such a complete understanding of its Nash equilibria has been given; for prior models only broad structural restrictions have been established.

Related Work. Networks play a major role in the economics literature. The structure and characteristics of such networks were first theoretically researched by Aumann and Myerson [4], but were empirically studied long before. For example, Granovetter [11] found that most residents of Massachusetts found their jobs through social contacts. For a recent and detailed review of social science and economics models see Jackson [12].

Remaining in the economics literature, but more directly related to our work, Corominas-Bosch [7] recently considered bipartite exchange economies, but focused on an iterated bargaining pricing mechanism (the Rubinstein mechanism), and analyzed the sub-game perfect equilibria of the game; network formation issues were not addressed. In other recent work, Kranton and Minehart [16] also considered bipartite exchange economies and network formation. In their models, buyer valuations are drawn from a known distribution, and the pricing mechanism used is that of a generalized English (ascending-bid) auction. Their main interests were the study of the efficiency of the formed networks and in showing that Nash equilibria networks are efficient; they also characterize Nash equilibrium structure for certain values of the edge cost. In contrast to these works, here we examine exchange equilibrium and provide a complete characterization of all Nash equilibrium networks.

Within computer science, most works have concentrated on network formation routing games, and the main interest has been the quality of the resulting equilibrium, as measured by the price of anarchy and the price of stability. We now survey most of these results.

Anshelevich et al. [3, 2] considered a network formation game in which each player or node is given a set of nodes to which she wishes to connect. Players are allowed to share the cost of an edge and thus may pay for remote edges. In the first work [3], any cost-sharing mechanism was considered and it was proven that there is a pure approximate 3-Nash equilibrium whose cost is that of the social optimum. An efficient algorithm to calculate an efficient 4.65-Nash was also provided. In the second paper [2] only a fair sharing mechanism that uses the Shapely value was considered. The price of anarchy in this setting is trivially $O(n)$, but they discovered that the *price of stability* was $O(\log n)$, and a matching lower bound was provided.

Fabrikant et al. [9], followed by Albres et al. [1], studied a game in which the goal of each player or node is to minimize the sum of distances to the other nodes and his edge costs, where the cost of each edge is α. The main results of these papers prove constant price of anarchy for almost every edge price α. A different variant of this model was studied by Corbo and Parkes [5] where the cost of an edge was shared equally by its endpoints; once again the main interest was in the price of anarchy, not in network structure.

Recently, Moscibroda et al. [17] studied a similar model with applications to peer-to-peer topologies. The goal of each player is to minimize the sum of stretches to other nodes and the edge costs. (The stretch is defined as the distance in the formed graph divided by an initial distance, which is decided according to the input metric). They also study the price of anarchy and the existence of pure Nash equilibrium.

Finally, Johari et al. [13] also considered a routing-based formation game. They considered a directed network, where each node wishes to send a given amount of traffic to other nodes. The cost function for a node/player v is composed from three components: the first is negative and is due to the edges purchased by v; the second is positive and is due to the nodes reachable from v; and the third is negative and is due to the amount of traffic that goes through v. The edges are bought by bilateral negotiation between the endpoints. The main results of [13] provide an equilibrium existence proof and a study of the equilibrium structure conditioned on the payoff function.

2 The Network Formation Game

We begin by reviewing the bipartite exchange economy model studied in [14, 15], and then extend this model to our network formation game.

2.1 Bipartite Exchange Economies

A bipartite exchange economy consists of a bipartite graph $G = (B, S, E)$, where nodes on one side of the bipartition represent buyers (B), and nodes on the other side of the bipartition represent sellers (S), and all edges in E are between B and S. There are two abstract commodities that, without loss of generality, we shall call *cash* and *wheat*. Buyer i has an infinitely divisible endowment of 1 unit of cash to trade for wheat; seller j has an infinitely divisible endowment of 1 unit of wheat to trade for cash. Buyers have utility x for x units of wheat and 0 utility for cash; similarly, sellers have utility x for x units of cash and 0 utility for wheat[1]. The semantics of the graph are as follows: buyer i can trade with seller j if and only if there is an edge between i and j.

Before describing the standard notion of equilibrium for this model, we note that it is a significant and deliberate specialization of the model first considered in [14], which among other features permitted varying initial endowments and utility functions, as well as an arbitrary number of commodities. As in [15], where the same specialization was adopted, our interests here are in the structures that arise purely from "network effects", as opposed to those arising from imbalances in supply and demand, variations in consumer utilities, and so on. We view the model adopted here — in which there is complete (initial) symmetry across all players — as the simplest model ensuring that any resulting network structure arises purely from the strategic aspects of the formation game, and we leave elaborations for future work.

We now describe our notion of exchange equilibrium for a bipartite exchange economy. Let ω_j^s denote the exchange rate (or price), in terms of cash per unit wheat, that seller j is offering. Similarly, let ω_i^b denote the exchange rate, in terms of wheat per unit cash, that buyer i is offering. Let x_{ij} denote the amount of seller j's wheat that buyer i consumes. A set of exchange rates, $\{\omega_i^b\}$ and $\{\omega_j^s\}$, and consumption plans, $\{x_{ij}\}$, constitutes an *exchange equilibrium for G* if the following two conditions hold [15]:

1. The market *clears*, *i.e.* supply equals demand. More formally, for each seller j, $\sum_{i \in N(s_j)} x_{ij} = 1$.[2] (The value of 1 on the right hand side represents j's endowment.)

2. For each buyer i, their consumption plan $\{x_{ij}\}_j$ is optimal. By this we mean that according to the consumption plan, buyers only buy from the sellers *in their neighborhood* offering the *best* exchange rate. That is, $x_{ij} > 0$ if and only if $w_j^s = \min_{s_k \in N(b_i)} w_k^s$.

We note that the role of buyers and sellers in a bipartite exchange economy is completely symmetric. Given buyer i's exchange rate, w_i^b, one can determine how much of buyer i's cash seller j consumes. Thus, one could equivalently define Item 1 above from the point of view of the buyers and Item 2 above from the point of view of the sellers. In this model, it turns out that an exchange equilibrium for G always exists if each seller has

[1] The exact form of these functions is irrelevant as each party has non-zero and increasing utility only for the "other" good.
[2] If i is a node in a graph $G = (V, E)$, then $N(i) = \{j | (i, j) \in E\}$.

at least one neighboring buyer (see [10, 14]). Furthermore, the equilibrium exchange rates are unique, and at equilibrium, if $x_{ij} > 0$ then $\omega_j^s = 1/\omega_i^b$.

Since each seller starts off with 1 unit of wheat and his exchange rate is in terms of cash per unit wheat, at exchange equilibrium each seller will earn exactly his exchange rate in dollars. Thus we will also call each sellers exchange rate w_j^s her *wealth*. Similarly, at exchange equilibrium each buyer will earn exactly his exchange rate in wheat, so we call the buyers exchange rate w_i^b her wealth as well. We say there is *no wealth variation* at exchange equilibrium of a bipartite exchange economy when the wealth of all of the sellers are equal and the wealth of all of the buyers are equal. We say there is wealth variation, at exchange equilibrium, when some buyers earn a different amount of wealth than other buyers and/or some sellers earn a different amount of wealth than other sellers. In a bipartite exchange economy where the number of buyers and sellers are equal, at exchange equilibrium the average wealth will be 1, so some player has wealth less than 1 if and only if some other player has wealth greater than 1.

Next we consider the graphical aspects of exchange equilibria. First, observe that in a bipartite exchange economy an exchange equilibrium not only determines the wealth of each player, but the consumption plan also determines on which edges trading takes place. We call the subgraph that consists of edges where trading occurred an *exchange subgraph*.

Definition 2.1 *Let $G = (B, S, E)$ be a bipartite exchange economy. Let $\{\omega_i^b\}$, $\{\omega_j^s\}$, and $\{x_{ij}\}$ be an exchange equilibrium, then the exchange subgraph of G is $G' = (B, S, E')$, where $E' = \{(i, j)|x_{ij} > 0\}$.*

In contrast to the exchange equilibrium wealth, the exchange subgraph need not be unique. We say that exchange subgraph G' is minimal if the removal of any edge from G' changes the exchange equilibrium wealths. Note that even when G is connected its exchange subgraph may be disconnected. We thus call the connected components of the exchange subgraph *trading components*. We say that a trading component is (m, k) if there are m buyers and k sellers, which will result in the wealth of each buyer in such component being k/m, and the wealth of each seller being m/k. Thus, wherever there is a wealth variation in G there are at least two trading components that have a different ratio between buyers and sellers in them.

In the bipartite exchange economy model described so far, the graph over which the buyers and sellers trade is exogenously defined. That is, the graph is fixed *a priori*, and then the players trade according to it. The main contribution of the work of [15] is to describe how the topology of the graph affects variation in price of the goods. The main contribution of this work is to make the formation of the graph endogenous to the game. That is, players are allowed to buy edges to other players, as opposed to having a topology imposed on them. We now give the formal definition of this new model.

2.2 The Network Formation Game

In this section we give a formal definition of the network formation game. This game consists of two sets of players, B and S, where $|B| = |S| = n$. The set B is defined as the buyer set, and the set S is defined as the seller set. As in the bipartite exchange economy we assume that each buyer starts off with an infinitely divisible endowment of 1 unit of an abstract good, which we call cash. Each seller starts off with an infinitely divisible endowment of 1 unit of another abstract good, which we call wheat.

The action of a buyer b_i is denoted $a_i^b \in \{0, 1\}^n$ and the action of seller j is denoted $a_j^s \in \{0, 1\}^n$. These actions encode which edges, if any, a player buys. An edge (b_i, s_j) is bought by player b_i only if $a_i^b(j) = 1$ and it is bought by s_j only if $a_j^s(i) = 1$. (At equilibrium, an edge (b_i, s_j) will be bought by b_i or s_j or neither, but not both.) A strategy is said to be pure if no player is randomizing over her actions; in this paper we study only pure strategies. Next, let $a = a_1^b \times \ldots \times a_n^b \times a_1^s \times \ldots \times a_n^s$ be the joint action of all the players. Let the set of edges that b_i buys be denoted $E_i^b(a) = \{(b_i, s_j)|a_i^b(j) = 1\}$, and let the set of edges that s_j buys be $E_j^s(a) = \{(b_i, s_j)|a_j^s(i) = 1\}$. The joint action of all the players defines a bipartite graph, $G(a) = (B, S, E)$

as follows. The nodes on one side of the graph represent the buyers and on the other side represent the sellers. The set of edges E are the edges that the players bought, or more formally: $E = \bigcup_{i\in[n],t\in\{b,s\}} E_i^t(a)$. Observe that every graph G defines a bipartite exchange economy. We call the price vector and consumption plan that form an equilibrium of the bipartite exchange economy an *exchange equilibrium*. This equilibrium will determine the wealth each player earns; the wealth that buyer b_i earns is denoted $\omega_i^b = \omega_i^b(G)$, and the wealth that seller s_j earns is denoted $\omega_j^s = \omega_i^s(G)$. The wealth each player earns will form the positive component of that players utility function. The negative component will be determined by how many edges each player buys. More formally, we define the utility functions of the players of type $t \in \{b, s\}$ in the network formation game as follows:

$$u_i^t(a) = u_i^t(a_i^t, a_{-i}^t) = \omega_i^t - \alpha|E_i^t|.$$

A joint action $a = a_1^b \times \ldots \times a_n^b \times a_1^s \times \ldots \times a_n^s$ is said to be a *Nash equilibrium* if for every player i we have $u_i^t(a_i^t, a_{-i}^t) \geq u_i^t(\hat{a}_i^t, a_{-i}^t)$ for every action $\hat{a}_i^t$. Since we only consider pure strategies for the players actions, we also only consider pure Nash equilibrium. Thus, each Nash equilibrium strategy a induces a graph, G, which we call an *equilibrium graph*.

Some important comments on this model are in order here. First, the utility functions u_i^t above specify the utilities or payoffs to the players of a standard *one-shot, simultaneous move game*: all players simultaneously choose the set of edges they wish to purchase, which in turn determines G and therefore the utility components ω_i^t. Second, it is important to note that there are *two* distinct equilibrium concepts we shall need to reason about. Our primary interest is in the (pure) Nash equilibrium of the game defined by the u_i^t, which is the network formation game. However, the definition of u_i^t itself involves another equilibrium quantity — namely, the wealth ω_i^t that i receives at *exchange* equilibrium in the *fixed* network G. For clarity we shall always refer to equilibria of the formation game given by the u_i simply as Nash equilibria, and to the latter notion as the *exchange equilibria* for a fixed G. Third, note that the u_i^t treat the initial purchase of edges and the exchange equilibrium wealths as taking place in the same "currency", which differs depending on the type of agent: buyers end with wealth measured in wheat, while sellers end with dollars. We can view this as modeling a central "edge banker" who is willing to extend credit in either currency to the players in order to allow the trade network to be built[3].

3 Summary of Main Results

In this section, we state and discuss our main results; proofs of the theorems are given in Section 4. Our first result relates the edge cost α to the minimum exchange equilibrium wealth in any Nash equilibrium graph.

Theorem 3.1 *Let G be a Nash equilibrium graph of the network formation game, and let ω_{min} be the minimum exchange equilibrium wealth in G of any player. Then $\alpha \geq 1 - \omega_{min}$, or equivalently, $\omega_{min} \geq 1 - \alpha$.*

Recalling that the average exchange equilibrium wealth is always 1 (since all endowments are equal), Theorem 3.1 states a natural limit on how much exchange equilibrium wealth variation can result from the formation game — the smaller the edge costs α, the more equitable these wealths must be. Great variation in wealths can only arise in the presence of high edge costs. The intuition behind this result is that a player of sufficiently low exchange wealth should be able to find another such player to trade with, with the resulting wealth gain more than covering the edge cost. The proof behind this intuition is somewhat subtle owing to the fragility of exchange equilibria — a small change to the underlying graph may cause large and distant changes to the exchange equilibrium.

[3]If desired, the notion of the edge banker can be made formal and endogenous to the game as a third player type with edges as initial endowments and equal utility for dollars and wheat.

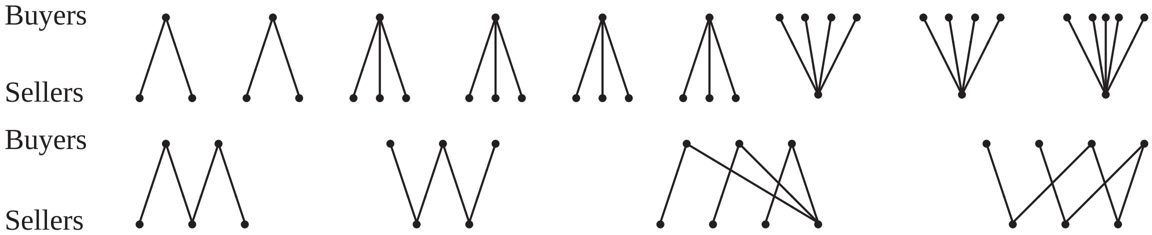

Figure 1: Top row: An example of an exploitation graph with $k = 2, \ell = 4$ and $n = 17$. Seller exchange equilibrium wealth values are $1/2, 1/3, 4$ and 5. Bottom row: An example of a balanced graph with $n = 14$. Seller exchange equilibrium wealth values are $2/3, 3/2, 3/4$ and $4/3$.

Theorem 3.2 *Let G be any bipartite graph, and let C be a trading component of G with buyer set B' and seller set S' such that $|B'| = m$ and $|S'| = k$, $m > k$. Then there exists a node s in S' and an edge e incident on s such that the removal of e from G decreases the exchange equilibrium wealth of s by at most most $1/k$. Furthermore, if G is a Nash equilibrium graph of the network formation game, then $\alpha \leq 1/k$.*

The second claim in Theorem 3.2 follows from the first by virtue of the fact that at Nash equilibrium, all of the edges in the trading component C must have been purchased by S' — since $m > k$, the buyers in B' are being "exploited" by the smaller number of sellers in S', and thus have better choices of edge purchases.

Now together Theorems 3.1 and 3.2 provide upper and lower bounds on the edge cost α in terms of the minimum exchange wealth and the possible trading component structure. It can be shown that together these bounds strongly constrain the possible Nash equilibrium graphs of the formation game, and that in turn the remaining possibilities can all in fact be realized, leading to a precise characterization of all Nash equilibrium graphs. Before stating our main theorem precisely, we define the following types of graphs.

- *Perfect Matchings.* The class of all perfect matchings between the buyers and sellers. In this class all exchange rates or wealths are equal to 1.

- *Exploitation Graphs.* These are graphs in which *every trading component* has a single party of one type (say, sellers) "exploiting" a (possibly much) larger set of parties of the other type, or vice-versa (a single buyer exploiting many sellers). The collection of such components must meet the constraint that there must be an equal number of buyers and sellers, but also a much stronger constraint on the number of possible different components that can be present simultaneously. More precisely, for any $k, \ell > 1$, let G be a graph consisting of the union of n_1 $(1, k)$-trading components, n_2 $(1, k+1)$-trading components, n_3 $(\ell, 1)$-trading components, and n_4 $(\ell+1, 1)$-trading components, where $n_1 + n_2 + n_3\ell + n_4(\ell+1) = n_1k + n_2(k+1) + n_3 + n_4$ (equal number of buyers and sellers). Note that in any such graph there may be at most 4 different (say) seller wealth values: $1/k, 1/(k+1), \ell$ and $\ell + 1$. Thus for large values of k or ℓ there is great wealth variation. The class of Exploitation Graphs consists of all such graphs G. See Figure 1 for an example.

- *Balanced Graphs.* While still permitting some inequality, these graphs are closer to the Perfect Matchings than to the Exploitation Graphs, in that wealth variation is strongly limited. More precisely, for any $k > 2$, let G be a graph consisting of the union of n_1 trading components that are either $(k-1, k)$ or $(k, k+1)$ and n_1 trading components that are either $(k, k-1)$ or $(k+1, k)$, $k > 2$. (Note that since the number of buyers is 1 less than the number of sellers in a $(k-1, k)$-trading component and a $(k, k+1)$-trading component, any mixture of n_1 such components is balanced by any mixture of n_1 $(k, k-1)$ and

$(k+1,k)$-trading components.) In such a graph there are again at most 4 different seller wealth values: $k/(k-1), (k+1)/k, (k-1)/k$, and $k/(k+1)$, but unlike in Exploitation Graphs, unbounded wealth variation is not possible, and for large k all wealths are nearly equal. See Figure 1 for an example.

Armed with these definitions, we can now state our main theorem, which provides a complete characterization of every Nash equilibria of our network formation game.

Theorem 3.3 *Let $NE(n,\alpha)$ be the set of all Nash equilibria graphs of the network formation game for a fixed population size n and edge cost α, and let NE be the union of $NE(n,\alpha)$ over all n and α. Then the set NE equals the union of classes Perfect Matchings, Exploitation Graphs, and Balanced Graphs defined above.*

As has been suggested, the proof that NE is contained in the stated union will follow from Theorems 3.1 and 3.2 above, while the proof that it contains the union will be shown by explicit construction which is deferred to Appendix A. We emphasize that Theorem 3.3 places very strong constraints on the Nash equilibrium graphs, and accordingly, on the nature of wealth variation. For instance, the characterization rules out certain exchange rates or wealths — $2/5$ is one example of an unattainable value. Wealth variation can essentially occur only in monopolistic form (the exploitation graphs).

Our results rely on one final structural characterization that is of independent interest, and concerns the "compactness" of Nash equilibrium graphs. More precisely, we show that a Nash equilibrium graph G cannot contain redundant edges — that is, the removal of any edge in G will change the exchange subgraph and the exchange rates or wealths. The intuition behind this theorem is that if redundant edges existed, the nodes that purchased them can remove them from the graph without effecting their wealth, and thus it is not a Nash equilibrium. Again there is some subtlety in the proof due to the aforementioned fragility of exchange equilibria. It is interesting to note that in other formation games, such as that in [1], cycles can exist at equilibrium, which can be seen as an analog of redundant edges in our formation game.

Theorem 3.4 *Let G be a Nash equilibrium graph of the network formation game. Then G is equal to it minimal exchange subgraph.*

4 The Analysis

4.1 Relating Topology to Equilibria in Bipartite Exchange Economies

In [15] the authors prove a few theorems that characterize the topology of graphs that give rise to wealth variation in a bipartite exchange economy. First we will state a theorem from [15] that provides sufficient and necessary conditions to wealth variation in bipartite exchange economy, where the number of buyers equals the number of sellers. Then we will extend it to economies where the number of buyers and sellers may not be equal. This will later help us compute the wealth of the sellers in each trading component, since in a given trading component, the number of buyers need not equal the number of sellers. For a bipartite graph $G=(B,S,E)$ if W is a set of nodes from one side of the bipartition, then $N(W)$ denotes the set of nodes connected by an edge to some node in W.

Theorem 4.1 *A necessary and sufficient condition for a bipartite exchange economy, $G=(B,S,E)$, to have* no *wealth variation among sellers, is that for all subsets $S'\subseteq S$, $|N(S')|\geq|S'|$.*

A symmetric argument can be made to show that there will be no wealth variation among the buyers, if for all subsets of buyers $B'\subseteq B$, $|N(B')|\geq|B'|$. By a trivial application of Hall's Theorem [6], we get the following result.

Input : $G_1 = (B_1, S_1, E)$ a bipartite exchange economy
Output : the trading components of G_1
$i = 1$;
repeat
    Let $U_i = \text{argmax}_{U \subseteq B_i} \frac{|U|}{|N(U)|}$;
    $C_i = \{U_i, N(U_i)\}$;
    $B_{i+1} = B_i \setminus U_i$, $S_{i+1} = S_i \setminus N(U_i)$;
    $E_{i+1} = E_i \setminus \{(u, v)|u \in U_i \text{ or } v \in N(U_i)\}$;
    $i = i + 1$;
    $G_i = (B_i, S_i, E_i)$;
until $B_i = \emptyset$;

Algorithm 1: This algorithm takes as input a bipartite exchange economy, G_1, and outputs the trading components, $C_1, ..., C_r$ of G_1.

Theorem 4.2 *There is no wealth variation in a bipartite exchange economy if and only if there is a perfect matching in the underlying graph.*

The extension the above theorem for unbalanced graphs can be found in Appendix B.

4.2 The Structure of Nash Equilibria of the Formation Game

The proofs of our main results use an algorithm for determining the trading components of a bipartite exchange economy. Algorithm 1 (see Figure) works by iteratively choosing the subset of buyers, $U \subseteq B$ that maximizes $|U|/|N(U)|$, outputs U and $N(U)$, removes them from the graph, and repeats. Intuitively the set of buyers, U, that maximizes this ratio will be getting fairly low wealth since there are many buyers connected to only a few sellers in $N(U)$. Furthermore, buyers not in U that are attached to the sellers in $N(U)$ will likely buy from other sellers since the price in $N(U)$ will be relatively high. We note that while there are more general and more efficient algorithms [8] for the equilibrium computation performed by Algorithm 1, its simplicity and properties (as we shall demonstrate) make it ideal for our structural analysis of the formation game Nash equilibria.

The proof of the following theorem and the analysis of the Algorithm can be found in Appendix C.

Theorem 4.3 *If Algorithm 1 is given any bipartite exchange economy G, then it will output all of the trading components of G (which comprises the exchange subgraph of G), along with the wealth of each buyer and seller in G. Furthermore, the connected components output by the algorithm are sorted according to the buyers' wealth in non-decreasing order, i.e., $\frac{|U_i|}{|N(U_i)|} \geq \frac{|U_{i+k}|}{|N(U_{i+k})|}$, for $k > 0$.*

A key implication of the above theorem is an analog for Theorem 4.1.

Corollary 4.1 *Let $C = (\tilde{B}, \tilde{S})$. Then C is a trading component if and only if for every subset $B' \subseteq \tilde{B}$, we have $\frac{|B'|}{|N(B')|} \leq \frac{|\tilde{B}|}{|N(\tilde{B})|}$.*

A symmetric claim holds for the sellers.

After showing that the algorithm indeed computes both the exchange equilibrium prices and the exchange subgraph in which they occur, we would like to prove the theorems stated in Section 3 using the algorithm's properties. Note that although the proofs rely on Algorithm 1, the statements are independent of the algorithms used to compute the exchange equilibria. The proof of Theorem 3.4, which states that the equilibrium graph equals its minimal exchange subgraph, is deferred to Appendix D.

Next we prove the theorem stating that α is lower bounded by 1 minus the minimum wealth.

Proof:[Proof of Theorem 3.1] Let $C_1, ..., C_r$ denote the connected components output by Algorithm 1 when its input is G (since G is an equilibrium graph of the network formation game, by Theorem 3.4 the connected components are G itself). Without loss of generality we assume that a buyer achieves the minimum wealth. Let $|U_1| = m$ and $|N(U_1)| = \ell$ and assume that $m > \ell$ so that $m = \ell + k$, $k > 0$, otherwise there is no wealth variation, and the bound is trivially satisfied. Also note that in C_r we have that $U_r = m'$ and $V_r = \ell'$, where $\ell' > m'$. This is because in G the number of sellers equals the number of buyers and by Theorem 4.3 the ratio in C_r is the minimum ratio. Now assume $u \in U_1$ connects to $v \in V_r$, and call the resulting graph G'. We now consider the set $U_1^- = U_1 \setminus \{u\}$. By definition we have that $|U_1^-| = m - 1$, and Theorem 4.3 implies that U_1 achieves the maximum ratio in G, so we must have that $|N(U_1^-)| = |N(U_1)|$. Thus we have $|U_1^-| = |N(U_1^-)| + k - 1$. More generally, by the maximality of U_1 in G we have that for every $W \subset U_1$, $|W| \leq |N(W)| + k - 1$. Our next step is to run Algorithm 1 on G'. Consider the iteration in which the last part of U_1^- is removed. Let W' be the part of U_1^- removed in all of the previous iterations, by the maximality of U_1 in G we have that $|W'| \leq |N(W')| + k - 1$, then $|U_1^- \setminus W'| \geq |N(U_1^- \setminus W')|$ and by Corollary 4.1 the last part of U_1^- will be removed with buyers' wealth at most 1. Thus, by Theorem 4.3, the buyers' wealth up to this point is at most 1. Buyer u could not be removed as a part of any previous set with wealth strictly smaller than 1 as u would have added one node to the set itself, and also would have added one node v to the neighbor set, and hence decrease the ratio. Therefore, we can assume that either u has not been removed up to this iteration, or it has been removed with wealth exactly 1. We next show that if it was not removed yet, it would also be removed with wealth 1. For any set W that does not contain u and that $\{v\} \subseteq N(W)$, we have $|W|/|N(W)| < 1$ (otherwise v would have had a higher wealth at G). Since after the removal of U_1^-, we have $|\{u\}|/|N(\{u\})| = 1$ and this is u's only edge remaining, u and v will be removed together and the wealth of each will be 1. Therefore, the wealth of u would increase by $1 - w_{min}$ by buying the edge, and since G is an equilibrium graph it implies that $\alpha \geq 1 - w_{min}$. □

Note that although the proof is referring to an equilibrium graph, we can deduce from the proof the fact that every node with wealth less than 1 can achieve price 1 by buying an additional edge. We also note that the following proposition regarding the identities of the players buying the edges follows from the same line of argument.

Theorem 4.4 *Let G be a Nash equilibrium graph of the network formation game. Then the exchange equilibrium wealth of each node which buys an edge is at least* 1.

We now go on and provide the upper bound theorem, which states that if an (m, k) trading component $(k < m)$ is part of an equilibrium graph, then α is at most $1/k$.

Proof:[Proof of Theorem 3.2] Let $C = (\tilde{B}, \tilde{S})$ be an (m, k)-trading component. We first find the strict subset of buyers inside the component with the maximum ratio and also its corresponding set of sellers.

$$\Gamma = \underset{\Gamma:\Gamma \subset \tilde{B} \text{ and } (|\Gamma| \neq m) \text{ and } W = N(\Gamma)}{\text{argmax}} \frac{|\Gamma|}{|W|}$$

Lemma 4.1 *Let $\beta = \frac{|\Gamma|}{|W|}$, then $\beta \geq \frac{m-1}{k}$*

Proof: Let us fix a set Γ' of size $m - 1$ then its neighbor set is $W' = N(\Gamma')$. Observe that the cardinality of W' is k, otherwise Γ''s ratio would be $\frac{m-1}{k-\ell} \geq \frac{m-1}{k-1} > \frac{m}{k}$, which by Corollary 4.1 contradicts the fact that (m, k) is a trading component. Applying Corollary 4.1 again implies either the component (Γ', W') has no wealth variation, or it has a subset with higher ratio (which has no wealth variation). □

We proceed and show the existence of a node in W that can remove one of its edges and decrease its wealth by at most $1/k$ as desired. Let us examine the nodes of W; we know that there exists at least one node $v \in W$ such that it has a neighbor, $u \notin \Gamma$, otherwise the component (m,k) is not connected. Now we consider the value of (v,u) to v. Let G' be the graph after the removal of (v,u). Now let $M_1 = (U_1, V_1), ..., M_K = (U_k, V_k)$ be the sets removed by the algorithm before v is removed (i.e $v \in V_{k+1}$). It remains to show that after the removal of these sets v's wealth is at least β, i.e. $\frac{|U_{k+1}|}{|V_{k+1}|}$ is at least $\frac{m-1}{k}$. In the next steps we use the following notation: $W_i = W \bigcap V_i$, $\Gamma_i = \Gamma \bigcap U_i$, $\bar{W}_k = \bigcup_{i=1}^{k} W_i$ and $\bar{\Gamma}_k = \bigcup_{i=1}^{k} \Gamma_i$.

Lemma 4.2 *For every* $k > 1$, $\frac{\bar{\Gamma}_k}{\bar{W}_k} \le \beta$

Proof: For every i, for every node $x \in \Gamma_i$ we have that in G, $N(\{x\}) \subseteq W$, therefore the only difference between $N(\{x\})$ with respect to G_i and $N(\{x\})$ with respect to G_1 are the nodes in W_j, $j < i$ (note that the edge (u,v) has no influence here). Thus, $N(\bar{\Gamma}_k) \subseteq \bar{W}_k$ in G, and by definition $(\bar{W}_k, \bar{\Gamma}_k)$ cannot attain a ratio larger than β. □

We now complete the proof of Theorem 3.2. Note that $|U_i|/|V_i|$ might be larger than β. Next we observe that if we partition Γ to two sets, $\Gamma \setminus \bar{\Gamma}_k$ and $\bar{\Gamma}_k$ such that $|\Gamma|/|N(\Gamma)| \ge |\bar{\Gamma}_k|/|N(\bar{\Gamma}_k)|$, then we must have $|\Gamma \setminus \bar{\Gamma}_k|/|N(\Gamma \setminus \bar{\Gamma}_k)| \ge |\Gamma|/|N(\Gamma)| = \beta$. Therefore, when v is removed, it is part of a set with ratio at least β which implies (as Algorithm 1 chooses the set that maximizes the ratio) that the set in which it is actually removed with has a ratio at least β as well (note that it is not necessarily $S \setminus \bar{S}_k$), and thus its wealth is at least β. Now the decrease in the wealth of v is at most $m/k - \beta \le 1/k$, which concludes the first part of theorem. Furthermore, if the trading component is part of an equilibrium graph of the network formation game, then by Theorem 4.4, v buys all of the edges incident on it and thus $\alpha \le 1/k$. □

We are finally ready to prove the first part of Theorem 3.3 which states that every Nash equilibrium is one of three stated types of graphs; in Appendix A we show that such equilibrium graphs do exist.

Proof:[Proof of Theorem 3.3, NE contained in three graph types.] Let C be an (m,k) trading component of the graph. We next show that only a few values of (m,k) can occur. By Theorem 3.1 we have that $\alpha \ge 1 - \frac{k}{m}$. By Theorem 3.2 we have that $\alpha \le 1/k$, combining these two inequalities we have that

$$1/k \ge 1 - k/m \Rightarrow m \ge k(m-k)$$

which holds only for $k = 1$ and $k = m-1$, and for $k = 2$ and $m = 4$. However, the $(4,2)$-trading component can only occur as a disjoint union of two $(2,1)$ trading components. Therefore, the only possible families of trading components are $(1,k)$ and $(k,k+1)$. First we show that $(1,k)$ trading component cannot coexist with $(\ell, \ell+1)$ trading components (unless $k = \ell = 2$). The first trading component implies that $\alpha \ge 1 - 1/k$, and the second implies that $\alpha \le 1/\ell$, and both can only simultaneously hold for $\ell = k = 2$ (and in such case the $(1,2)$ trading component can be considered as either an exploitation graph or a balanced graph).

Now consider the case where $(1,k)$ trading component exists along with $(1, k+\ell)$ for $\ell \ge 2$, and let u be the sole buyer in the $(1,k)$ trading component. Now u can buy an edge to a node v in the $(1, k+\ell)$ component; it is easy to see that now v will trade with u, and v will earn wealth $1/(k+1)$, rather than trading inside the $k+\ell$ component and earning wealth $1/(k+\ell)$. Therefore, it cannot be an equilibrium graph of the network formation game. It sill remains to show that $(k, k+1)$ cannot coexist with $(\ell, \ell+1)$. Let us see what are the restrictions imposed by such component. By Theorem 3.2 and Theorem 3.1, we have that

$$\frac{1}{k+1} \le \alpha \le \frac{1}{k}$$

This immediately implies that only consecutive trading components, i.e, $(k-1,k)$ and $(k,k+1)$, can coexist. □

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A Construction of Equilibrium Graphs

A.1 Exploitation(k, l) Graphs

In this section we show that there are Exploitation(k, ℓ) which are Nash equilibria of the network formation game. We start by providing a technical lemma.

Lemma A.1 *If $\alpha > 1-1/\ell$ then at Nash equilibrium of the network formation game, no seller in a Exploitation(k, ℓ) graph would buy an edge to a buyer of degree ℓ or $\ell + 1$.*

Proof: Let s be a seller, and let b be a buyer of degree ℓ or $\ell+1$. At Nash equilibrium of the network formation game, b's wealth is at most $1/\ell$. Since the players are rational, the only way for trade to occur over a (b, s) edge is if s offered a price lower than $1/\ell$. Since $\alpha > 1 - 1/\ell$, s would only decrease her utility by buying an edge to b. □

Now we are ready to show that Exploitation(k, ℓ) graphs can be equilibria graphs of the network formation game.

Lemma A.2 *If $\alpha > 1 - 2/(\max(k + 1, \ell + 1))^2$, then any Exploitation$(k, \ell)$ graph where nodes with degree k or $k + 1$ and ℓ or $\ell + 1$ buy all the edges incident on them, is a Nash equilibrium of the network formation game.*

Proof: Let s be a seller with degree k or $k + 1$. By Lemma A.1 we know that at Nash equilibrium of the network formation game s will not buy any edges to buyers of degree ℓ or $\ell + 1$. Next, let $w_1, \ldots, w_{k'}$ be the buyers attached to s in a Exploitation(k, ℓ) graph. Now say s bought a set of edges that did not contain all of the w_i. Then, since $\alpha < 1$ and no other players are connected to the w_i, s could increase its utility by buying edges to the those unconnected w_i. Thus we can assume, at Nash equilibrium of the network formation, s buys all the edges to the w_i. Next, we show at formation equilibrium s does not buy edges to other degree 1 nodes.

By the market clearing condition, the wealth of each of the w_i is either $1/k$ or $1/(k + 1)$, and s wealth is either k or $k + 1$. If s has wealth $k + 1$ and it buys edges to buyers that have wealth k, these buyers have no incentive to switch to s so it would not be rational for s to buy an edge to such a buyer. Next, if s wealth is k and it buys an edges to a buyer b that is also getting price k, b would not buy from s for the following reason. Assume that if (b, s) is not an edge in the graph b buys from s', but when (b, s) is an edge in the graph b buys from s. Then at market clearing s would offer price $k + 1$ and then s' would offer price $k - 1$. Thus it would not be rational for b to switch sellers. Finally, if s wealth is $k + 1$ and a buyer b is getting price k from s', s will not buy the edge to b. If s did, then b will split its good evenly between s and s'. Since the cost of an edge is $\alpha > 1 - 1/(\max(k + 1, \ell + 1))^2$, where $k, \ell > 1$ and this edge only increased the utility of s by $1/2$, s would not buy this edge.

Next, let s be a seller of degree 1. Again, by Lemma A.1 we know that at equilibrium s will not buy any edges to buyers of degree ℓ or $\ell + 1$. So, all we have to show is that at equilibrium s will not buy any edges to buyers of degree 1. Consider the result of s buying a edges to a set of buyers B, where $|B| = m$. If $m > \lfloor(\ell + 1)/2\rfloor$, s wealth would only be $\lfloor(\ell + 1)/2\rfloor$. So $m \leq \lfloor(\ell + 1)/2\rfloor$, in which case s wealth would be m and pay $m\alpha$ for the edges to B. Observe that $m(1 - \alpha) \leq (\ell + 1)(1 - \alpha)/2 < 1/(k + 1)$. Thus, buying these m edges would only decrease the utility of s.

Thus, we have shown that at Nash equilibrium of the network formation game sellers would buy only those edges designated by the Exploitation(k, l) graph. The case for buyers is entirely symmetric. □

A.2 Balanced Graphs

In this subsection we show that balanced graph are equilibria graphs of the network formation game for appropriate values of α. We start by showing that any $(k, k + 1)$ minimal trading component can be part of a balanced graph. Note that in comparison to the previous trading components a $(k, k+1)$ components can differ from each other. We start by characterizing every minimal $(k, k+1)$ trading component.

Lemma A.3 *Let $C = (\tilde{B}, \tilde{S})$ be a $(k, k+1)$ minimal trading component then the degree of each $b \in \tilde{B}$ is exactly 2.*

Proof: Suppose for the sake of a contradiction that there exists a node, $b \in \tilde{B}$ with degree 3 in C. Let s_1, s_2, s_3 be its neighbors in $\tilde{S}$. By the minimality of the trading component we have that there exists three subsets S_1, S_2, S_3, such that $s_i \in S_i$ and that $|S_i| - 1 = |N(S_i) \setminus \{b\}|$ and that for every $S' \subset S_i$ it does not hold (if such do not exist then we can remove the edge and the trading component will not be effected, which is a contradiction to its minimality). Let $\bar{S}$ be the union of S_1, S_2 and S_3 excluding $\{s_1, s_2, s_3\}$. Since C is trading component, then by Corollary 4.1 for every subset S' of $\bar{S}$, we have $|S'| \leq |N(S')|$, and thus there exists a perfect matching between $\bar{S}$ and $N(\bar{S})$, and their cardinality is identical, denote it by ℓ. Now consider the set $\bar{S}$ with $\{s_1, s_2, s_3\}$, its cardinality is $\ell + 3$ however, the cardinality of $|N(\bar{S} \bigcup \{s_1, s_2, s_3\})|$ is $\ell + 1$. This is due to fact that if s_1 for instance will add two nodes to $N(\bar{S})$, then we will have $|S_1 \setminus \{s_1\}| = |N(S_1 \setminus \{s_1\})| + 2$; this yields an higher ratio than C's ratio, which contradicts the fact that C is a trading component by Corollary 4.1. Thus the degree of b at most 2. Note that the degree of b cannot be 1 as its wealth is strictly larger than 1. □

Using this characterization for every $(k, k+1)$ minimal trading component, we can show that balanced graphs are Nash equilibria of the network formation game for specific values of α.

Lemma A.4 *If $1/(k+1) \leq \alpha \leq 1/(k)$ then any balanced graph consisting n_1-$(k, k+1)$ trading components and n_1-$(k+1, k)$ trading components is an equilibrium graph of the network formation game.*

If $\alpha = 1/(k+1)$, then balanced graph consisting n_1 $(k-1, k)$ or $(k, k+1)$ trading components and n_1 $(k, k-1)$ or $(k+1, k)$ trading components is an equilibrium graph of the network formation game.

Proof:(sketch) We sketch the proof for the first case and omit the similar second case. Let us consider the strategy of players in a $(k, k+1)$ connected component, C, where $\tilde{B}$ is the buyer set and its cardinality is k and $\tilde{S}$ is the seller set with cardinality $k+1$. Before going over all possible cases, we provide the following fact:
(A) For a graph G, if the lowest wealth obtained by buyers (sellers) is β, then for a graph G', such that $G \subseteq G'$, the wealth of any seller(buyer) is at most $1/\beta$.

We first consider every possible deviation of $b \in \tilde{B}$, whose current utility is $\frac{k+1}{k} - 2\alpha$ by Lemma A.3:

1. **Removing one edge:** By the minimality of the trading component, if b removes an edge to s then there exists a subset $S' \subset \tilde{S}$, which now satisfies $|S'| = |N(S')| - 1$. If Alg. 1 input is $G \setminus \{(b, s)\}$ then S' is removed first, and since for every other subset W of $\tilde{S}$ that $|W| \leq |N(W)|$, and thus there is no wealth variation and b's new wealth will be 1. Therefore b gains α from removing the edge but her wealth decreases by $1/k$ and will have no incentive to deviate.

2. **Removing two edges:** By Lemma A.3 b has exactly two edges, and the removal of both will make her utility 0, which is clearly less than its previous utility $\frac{k+1}{k} - 2\alpha$.

3. **Buying additional edges:** By fact(A) we have that b cannot have wealth larger $\frac{k+1}{k}$ by adding edges to G.

4. **Removing one edge and buying additional edges:** By the previous case we have that after removing one edge the wealth of b is 1. Thus the trading component (without the additional edges) are $(\ell, \ell+1)$ for $\ell < k$, and a perfect matching component of size $(k - \ell)$ (both due to the decomposition of C), and the rest are the $(k, k+1)$ and $(k+1, k)$ trading components. Adding edges to any of $(k, k+1)$

or $(k+1,k)$ cannot yield a wealth higher than $\frac{k+1}{k}$ as $\frac{k}{k+1}$ is the lowest wealth before, and thus the b has no incentive to buy one. Now it is easy to see that buying edge into the $(\ell, \ell+1)$ will form the $(k, k+1)$ trading component again, and buying additional edges (to either the $(\ell, \ell+1)$ component or to the perfect matching component will have no influence on the prices). Therefore, the node price will be at most $\frac{k+1}{k}$ and it will by at least two edges, and thus will have no incentive to deviate.

5. **Removing two edges and buying additional edges:** By Lemma A.3 and the minimality, it is not hard to see that after removing two edges,C is decomposed to b (isolated), (ℓ_1, ℓ_1+1) and (ℓ_2, ℓ_2+1) trading components (note that ℓ_2 can be 0 and that $\ell_1+\ell_2 = k-1$). The other trading components are the former $(k, k+1)$, $(k+1, k)$ trading components. Once again buying edges into the $(k, k+1)$, $(k+1, k)$ trading components cannot help as the maximal price that can that be obtained is $\frac{k+1}{k}$ and at least two edges should be bought to achieve it. Now buying one or two edges to the the smaller components (that are created from the disconnection) is identical to the previous cases.

Now let us consider a node $s \in \tilde{S}$ possible deviation. Note that s current strategy is not buying any edge and thus her utility is $\frac{k}{k+1}$. We first note that forming edges to nodes b with current wealth larger than 1 is never beneficial as b will prefer not to trade with s, and they will not effect the trading components.

1. **Buying a single edge:** The edge must be to a node with price $\frac{k}{k+1}$, and by similar arguments to Theorem 3.1, s wealth will be 1, and thus its new utility is $1-\alpha \le 1-\frac{1}{k+1} = \frac{k}{k+1}$, which is less than her current utility.

2. **Buying** 2 **edges:** In such case one can see that s will be now part of a $(k+2, k+1)$ trading component (as a node with wealth higher than 1), however its utility will be $\frac{k+2}{k+1} - 2\alpha$ which is smaller than $\frac{k}{k+1}$ for $\alpha \ge 1/(k+1)$.

3. **Buying at least three edges:** By fact(A) the wealth is bounded by $\frac{k+1}{k}$ and thus the increase in the wealth is $\frac{k+1}{k} - \frac{k}{k+1}$ which is at most $2/k$. The edges' cost on the other hand is $\ell\alpha$, which is at least $3/(k+1)$, which is always larger than $2/k$.

□

A.3 Completing the Proof of Theorem 3.3

We are now finally ready to complete the proof of Theorem 3.3, by showing that the set Nash equilibria graphs of the network formation game contain graphs which are (1)Prefect matching, (2)Exploitation (k, ℓ) and (3)Balanced graphs.

Proof:[Proof of Theorem 3.3 part(B)] Consider a perfect matching graph. Any node that does not buy an edge cannot change her wealth by buying an edge, as it will still be a perfect matching. If a node that buys an edge, will deviate to a strategy in which it removes her edge and buy ℓ edges, it will have a wealth of $\frac{\ell}{\ell+1}$ and thus will have no incentive to do so. If a node, which buys an edge, buys additional ℓ edges, it will have no influence on the prices. By Lemma A.2, (k, ℓ)Exploitation are Nash equilibrium for the appropriate α's, and by Lemma A.4 balanced graphs are Nash equilibrium of the game for appropriate α's. □

B Extension of Theorem 4.2 into unbalanced graphs

Theorem 4.2 provides a necessary and sufficient condition to have no wealth variation in a graph where the number of sellers equals the number of buyers. Next, we will extend that theorem to graphs where the number of buyers does not equal the number of sellers. First we will provide a construction that will transform a graph with an unequal number of buyers and sellers to a graph with an equal number of buyers and sellers. Then, we show that the transformed graph has a perfect matching if and only if the original graph has no wealth variation.

Definition B.1 *Let $G = (B, S, E)$ be a bipartite graph such that $|B| = n$ and $|S| = m$. Its τ-*balanced graph *$G' = (B', S', E')$ is constructed as follows. For each $b_i \in B$ make m/τ copies in B', call them $b^i_1, \ldots b^i_{m/\tau}$, and for each $s_j \in S$ make n/τ copies of it in S', call them $s^j_1, \ldots s^j_{n/\tau}$. Finally, for each edge $(b_i, s_j) \in E$, add edges to E' to form the complete bipartite graph between $b^i_1, \ldots b^i_{m/\tau}$ and $s^j_1, \ldots s^j_{n/\tau}$.*

Lemma B.1 *Let $G = (B, S, E)$ be a bipartite graph such that $|B| = n$ and $|S| = m$. Let $\tau > 0$ be the maximum number such that each element of an exchange equilibrium consumption plan $\{x_{ij}\}$ can be represented as $k\tau$ for an integer k.* [4] *Let $G' = (B', S', E')$ be the τ-balanced graph of G. Then there exists an exchange equilibrium consumption plan for G, where the buyers all earn wealth m/n and the sellers all earn wealth n/m, if and only if G' has a perfect matching.*

Proof: ($\leftarrow$ direction) If G' contains a perfect matching, then an exchange equilibrium consumption plan for G can be defined as follows. For every edge of (b^i_k, s^j_l) of the perfect matching, $b^i_k \in B'$, $s^j_l \in S'$ add τ/m units of cash going from b_i to s_j, and τ/n units of wheat going from s_j to b_i. By construction, every buyer $b_i \in B$ has m/τ copies in G', and since there is a perfect matching in G', the amount of wheat earned by b_i is m/n as desired. Similarly, every seller $s_j \in S$ has n/τ copies in G' and since there is a perfect matching in G', the amount of cash earned by s_j is n/m as desired.

($\rightarrow$ direction) If there is an exchange equilibrium consumption plan where all buyers earn wealth m/n for G, then each edge (b_i, s_j), $b_i \in B, s_j \in S$ with $k\tau$ units of cash going from b_i to s_j, can be partitioned into km distinct edges between the sets of nodes $\{b^i_k\}_{k=1}^{m/\tau} \subseteq B'$ and $\{s^j_l\}_{l=1}^{n/\tau} \subseteq S'$. These edges in G' can be viewed as carrying τ/m units of cash from a b^i_k to a s^j_l and they will form the perfect matching edges. Now let us count how many edges each node $b_i \in B$ induces. By the market clearing condition, the total expenditure of b_i is 1 unit of cash. Thus b_i induces m/τ edges which equals the number of corresponding $b^i_k \in B'$.

For $s_j \in S$ we have that its incoming flow is n/m units of cash. Furthermore, every edge incident on s_j was split into km copies each carrying τ/m units of cash from b^i_k to s^j_l. Thus there must be n/τ edges incident on the $\{s^j_l\}_{l=1}^{n/\tau}$. Therefore, each edge can be matched with a unique pair of nodes in G' and form a perfect matching. $\Box$

C Algorithm Correctness Proof

In the next lemma we show if the players in a bipartite exchange economy trade within the trading components output by Algorithm 1, then there will be no wealth variation within each trading component.

Lemma C.1 *If Algorithm 1 is run on a bipartite exchange economy $G = (B, S, E)$, and the players trade within the trading components output, $C_1 = \{U_1, N(U_1)\}, \ldots, C_r = \{U_r, N(U_r)\}$, then at every C_i will have no wealth variation, furthermore, the wealth of each seller will be $|U_i|/|N(U_i)|$, and the wealth of each buyer will be $|N(U_i)|/|U_i|$.*

Proof: Let $\tau > 0$ be the maximum number such that each element of an exchange equilibrium consumption plan $\{x_{ij}\}$ can be represented as $k\tau$ for an integer k. Let $H = (U_i, N(U_i), E_i)$, where $E_i = \{(u, v) | u \in U_i, v \in N(U_i)$, and $(u, v) \in E\}$. Let $H' = \{U'_i, V'_i, E'_i\}$ be the τ-balanced graph of H. Assume for the sake of contradiction that there is wealth variation in H. Then, by Lemma B.1 there is no perfect matching

[4] Since the utilities and endowments of the players are rational, the values of the consumption plan are also rational [8]. Thus, such a τ must exist.

in H'. Next, by Theorem 4.2 if there is no perfect matching in H', then there is a set $W \subset U'_i$, such that $|N(W)| < |W|$. By the construction of H' we can assume, without loss of generality, that if one of the nodes corresponding to b_i in H' is in W, then all of the nodes corresponding to b_i in H' are in W. More formally, if $b^i_k \in W$ then $\{b^i_k\}_{k=1}^{m/\tau} \subseteq W$. Now let $R(W) = \{b_i | b^i_k \in W \text{ for some } k\}$, then we get the following. (The left most equality comes from the construction of H'.)

$$\begin{aligned} \frac{|R(W)|}{|N(R(W))|} &= \frac{\tau|W|/|N(U_i)|}{\tau|N(W)|/|U_i|} = \frac{|W|}{|N(W)|}\frac{|U_i|}{|N(U_i)|} \\ &> \frac{|U_i|}{|N(U_i)|} \end{aligned}$$

This contradicts the fact that $|U_i|/|N(U_i)|$ has the maximum ratio at G_{i-1}. Thus there is no wealth variation in H. Since there is no wealth variation in H at exchange equilibrium, and because of the market clearing condition, the wealth of each seller must be $|U_i|/|N(U_i)|$, and the wealth of each buyer must be $|N(U_i)|/|U_i|$ □

This next lemma establishes the fact that as Algorithm 1 runs, the ratio of the size of the subsets of buyers to the size of their neighbor sets, $|U_i|/|N(U_i)|$, is non-increasing. This is essential to the proof of correctness of our algorithm, because the algorithm assumes that U_i and $N(U_i)$ will form a trading component, and this result shows that neither set will have better trading opportunities.

Lemma C.2 *For any run of Algorithm 1, $|U_i|/|N(U_i)| \geq |U_{i+k}|/|N(U_{i+k})|$ for $k > 0$.*

Proof: Assume for the sake of contradiction that the lemma does not hold, then there exists two consecutive sets such that $|U_i|/|N(U_i)| < |U_{i+1}|/|N(U_{i+1})|$. Consider the set $U_i \bigcup U_{i+1}$ during a run of Algorithm 1 just before U_i and $N(U_i)$ were removed from the graph.

$$\frac{|U_i \bigcup U_{i+1}|}{|N(U_i \bigcup U_{i+1})|} \geq \frac{|U_i| + |U_{i+1}|}{|N(U_i)| + |N(U_{i+1})|} > \frac{|U_i|}{|N(U_i)|},$$

which contradicts the maximality of U_i. □

As consequence of lemmas C.1 and C.2, the proof of Theorem 4.3 follows.

D Proof of Theorem 3.4

Proof:[Proof of Theorem 3.4]

Let G' be the exchange subgraph of G. Assume for the sake of contradiction that G' is a strict subgraph of G. There are two types of edges which can be in G and not in G'. The first is edges inside a connected component, and the second is edges connecting two connected components. Let C_i be the first connected component outputted by Alg. 1 that has redundant edges and assume wlog that $i = 1$.

We first deal with the redundant edges inside C_1. Now as long as U_1 edges $N(U_1)$ remain unchanged and there is no proper subset, W of U_1 with larger ratio (i.e. $|W|/|N(W)| \leq |U_1|/|N(U_1)|$), then Alg. 1 will still output U_1 as the first connected component; thus any edge that does not violate these requirements can be removed without effecting the exchange subgraph, and since an edge price is positive G cannot be an equilibrium graph of the network formation game.

After dealing with edges of the first type, we proceed to the second type. By definition there are no edges going out of U_1 outside the set $N(U_1)$, and we remain to deal with edges inside/outside $N(U_1)$. By Lemma C.2 we have that for every set in G_2, we have $|S|/|N(S)| \leq |U_1|/|N(U_1)|$. Adding the edges that removed while moving to G_2, imply that in G_1 for every set we have that $|S|/|N(S)|$ is only smaller (as $N(S)$ might be larger), and thus U_1 remains the maximum in G_1 and the edges are redundant in G_1 and since they do not appear in G_i $i \geq 2$ they are redundant and can be removed without effecting the exchange subgraph. □