THE EFFECTS OF NETWORK TOPOLOGY ON STRATEGIC BEHAVIOR

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Dedication

to Mom, Dad, and Bhai

and to Miranda

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ABSTRACT

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The central question this thesis addresses is: if players are arranged in a network, and they are strategically interacting only with other players in their local neighborhood, how does the topology of the network affect the outcome of the interaction? We answer this question by combining techniques from computer science, economics, and sociology.

First, we introduce a graphical market model consisting of a bipartite network with buyers on one side and sellers on the other. Trade can only occur between a buyer and a seller if they are adjacent. We characterize when there will be variation in equilibrium wealth in terms of the network topology. Furthermore, we quantify the equilibrium wealth variation in social networks in terms of the degree distribution. Both of these results show that the network topology strongly affects the equilibrium behavior in this model. We also analyze a similar model where the players must buy the edges and trade according to the network that was purchased. We give a complete characterization of the equilibrium networks in this model.

Second, we assess a model of evolutionary game theory over networks. Evolutionary game theory has been used to model biological and social interactions where the dynamics are more imitative than optimizing. For two broad classes of graphs, we characterize which strategies could be played by a large fraction of the population that would guarantee they could not be overrun by any small mutant invasion. Here again, the topology of the networks under study had a direct impact on the structure of equilibria.

The final main component of this work is experimental in nature. A group of undergraduates was instructed to solve the graph coloring problem. Each one used our software system to control the color of one node in a network, and their objective

was to arrive at a valid coloring. We varied the topology of the network by using different generative models from social network theory. One key finding of this experiment was that the group could color networks with low diameter much faster than networks with high diameter — showing that network topology impacted group behavior.

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Chapter 1

Introduction

This thesis uses techniques from computer science to explore the intersection of social network theory[1] and economics. The field of social network theory primarily focuses on measuring the topology of naturally occurring networks and then inventing generative models that output networks of similar structure (see for example [103, 7]). This field directs most of its attention toward networks that describe the interaction of large populations of humans and produces models that are purely structural. That is, social network theory models patterns of links between nodes in networks rather than attempting to model the behavior of the people that these nodes represent. By turning to the field of economics and, specifically game theory, which describes many models of interaction, we can incorporate behavioral dynamics into our structural understanding of networks. Game theory analyzes models of fully rational agents acting in their own best interests, models of collusion and cooperation between agents, and even behavioral models based on experiments with actual human subjects (see for example [88, 17]). The overwhelming majority of these models, however, do not consider network structure. They assume that all players can

[1]Since this field is relatively new, its name has not yet been codified. In this thesis we use the term social network theory to refer to the study of interactions between members of a large population; it has alternately been referred to as network science [22].

interact with and directly affect all other players. Combining the interaction models of game theory with the networks models of social network theory yields a much richer, more accurate model of human behavior.

Most of the work in social network theory involves the measurement and observation of social networks. For example, work in this field studies statistical properties of networks such as the degree distribution of the nodes, the average number of links nodes maintain, the density of the edges, and the diameter of the entire network. These types of statistics are often measured in real networks like the World Wide Web, phone call graphs, e-mail networks, on-line communities, and co-authorship networks (see for example [70, 50, 6, 71]). Once these types of statistics are found in a variety of real world networks, the field creates generative models that output networks with similar statistical properties. Two such models, which we discuss later in this thesis, are called preferential attachment and small-world [95, 8, 103]. All of these generative models have a few characteristics in common. They usually build the graph sequentially by adding nodes and/or edges one at a time. Usually these new nodes and edges are added via a stochastic dynamic. For example, in the case of preferential attachment a new node attaches to an existing node with probability proportional to that existing node's degree. While these models often output networks with similar statistical properties as those seen in nature, these stochastic dynamics do not *explicitly* model the behavior of the players within the network. That is, these dynamics model the structure of the network formation but not the dynamics of the interactions that occur over these networks.

The field of game theory, on the other hand, provides many models of interaction. Most game theoretic models consist of n players where the combined actions of all players determines the payoffs to each individual player. Thus each player takes into account the actions the other players might select when deciding his own action. The field considers models where players interact in their own self interest by choosing actions with the aim of maximizing their own payoff. Some models also allow for coalitions to form among groups of players [88]. Coming even closer to the real-world behavior of people, the field of behavioral game theory derives models based on observations from human subject experiments involving strategic games [17]. Game theory considers a variety of different mechanisms to determine the payoffs to the players. Many models use an abstract, multidimensional matrix to determine the payoff for each player. Others models, such as those discussed in this thesis, use a market mechanism or a model based on evolutionary dynamics where a higher payoff represents a higher reproductive fitness [106]. The vast majority of these models only consider settings where the action of each player can directly affect the payoff of every other player. Put another way, in these models each player can interact with every other player. As a result, these models do not consider the constraints that networks impose on the interaction of the players.

There is a clear way that these two fields can inform one another: the game theoretic models of interaction can be applied over networks, especially those generated by the models proposed by social network theory. In these models players would still act strategically, but they would only be able to interact with the other players in their local neighborhood. This would allow for the analysis of the behavior of the players over realistic networks. Such interdisciplinary models provide an important contribution because behavioral analysis is generally missing from the social network theory models, and the study of realistic networks is largely lacking in the game theory literature. Furthermore, the theoretical and empirical analysis of models of interaction over networks would allow for the study of many fascinating sociological questions on a large scale. For example, one could study how people behave differently when arranged in different networks, how people organize themselves into communities, and how innovations spread across networks.

The field of computer science is in a unique position to contribute to the synthesis of social network theory and game theory. From an empirical point of view, our field has already developed many techniques appropriate for studying massive data sets that inspire the generative models of social network theory. Techniques from streaming algorithms, sub-linear time algorithms, and external memory algorithms can be used to study networks as big as the World Wide Web or the phone call graph (see [82, 101] for excellent surveys of these areas). Additionally, techniques from machine learning can be used to predict how people behave differently in different networks or how these networks change over time. These types of analyses would aid in the development of behavioral models over networks. Developing such models is important because they often elucidate the governing dynamics of the players. Analyzing these dynamics would lead to a better understanding of how changes to local incentives and local interactions effect the behavior of a population as a whole. Techniques from computer science can also help explore these models from a more theoretical perspective. Algorithms for computing equilibria and best responses can be used to analyze the equilibria of these network-based models (see [92], especially [59]). Also, simulation techniques can be used to study the behavior of the players in theoretical models. Finally, the broad literature on graph theory can be used to help understand the topological aspects of large scale networks [13, 20].

The intersection of social network theory and game theory, when studied from a computer science point of view, provides a timely new research area teeming with unexplored, well-motivated research directions. It is only recently, with the advent of the Internet, that it has been feasible to gather large data sets to inform this study. It is also only very recently that scientists have had the computational power readily available to make the study data sets of these magnitudes feasible. Thus the intersection of these three fields is currently ripe for exploration, and the work in this thesis sits squarely at their nexus. Here we will combine economic and game theoretic models of interactions with networks which are inspired by social network theory. Throughout this thesis we use techniques from computer science to explore these models. This new area is, however, far too broad to be fully covered by any one thesis. Thus we begin its exploration by addressing one fundamental question throughout this work: if players are arranged in a network, and they are strategically interacting only with other players in their local neighborhood, how does the topology of the network affect the outcome of the interaction? Since this question is motivated by a variety of fields, we use a variety of techniques to answer it. First and foremost we will use formal analysis of the theoretical models we introduce. We will often support these theorems with simulations on both artificially produced and real world data. These simulations will show how tight the bounds proved in these theorems are and how close a large, but finite, population gets to a predicted asymptotic behavior. Finally, we will also use experiments on actual human subjects and perform statistical analysis of the data gathered from these experiments. All of these models, simulations, and experiments and their respective analyses will serve to elucidate the relationship between topology and behavior, thereby speaking to the productive juncture of insights from social network theory and game theory.

Now we go on to give a brief overview of the models and experiments introduced in this thesis and show how their analysis exhibits the effect of network topology on behavior and outcome. First, we analyze the effect of network topology on the equilibrium payoff of players in a market. To do this, we combine a model of markets from economics with a network generation model from social network theory. We consider a bipartite network with buyers on one side and sellers on the other. Buyers start with an initial endowment of one unit of an infinitely divisible abstract good, which we call cash. Sellers start with an initial endowment of one unit of an infinitely divisible good, which we call wheat. We assume that buyers wish to trade their cash for as much wheat as possible, and that sellers wish to trade their wheat for as much cash as possible. Furthermore, trade can occur between a buyer and a seller only if they are attached by an edge. We examine how the topology of this network affects the equilibrium payoff of the players in both general networks and networks produced via preferential attachment, which is a model from social network theory. We are particularly interested in how the statistical structure of such networks influences global economic quantities such as wealth variation. We also show how this graphical model allows for the efficient approximation of a player's equilibrium wealth. Our findings, which were originally published in [56], combine theoretical analysis, simulation, and experiments on real world international trade data.

Second, we analyze a model similar to the one previously discussed. In the previous model the trade network was *exogenously* defined. Here we analyze a similar model where the trade network is *endogenously* defined. That is, buyers and sellers still trade according to a network, but in this case the edges must be purchased. Once the edges are purchased, then buyers and sellers trade according to the graph that was paid for. We give a complete characterization of the equilibrium graphs that can form and the equilibrium payoffs of the players when the construction of the network is part of the game itself. This allows us to analyze how the topology of the network affects the payoff of the players. This is theoretical work that originally appeared in [37].

Third, we analyze a model of evolutionary game theory over networks. Evolutionary game theory has become a plausible model not only for biological interaction but also for economic and social interactions in which certain dynamics are more imitative than optimizing [98]. The model consists of organisms arranged in a network in which each organism plays the same two-player game with each of its neighbors. Each organism is of one of two types: incumbent or mutant. Incumbents all play one strategy and mutants all play another strategy. These conditions result in each organism earning a payoff in terms of their reproductive or evolutionary fitness. For two broad classes of graphs, we characterize the strategies incumbents could play that would guarantee they have a higher fitness than the mutants in any small invasion. Such strategies ensure that the incumbent population could not be overrun by any small mutant invasion. Thus we characterize how the topology of the network affects the resilience of a strategy to competing strategies. This theoretical work, which first appeared in [62], integrates game theory, as a model for social interaction, with graph theory.

Finally, we explore the relationship between network topology and behavior in a setting that combines behavioral game theory and social network theory. In this case the players are actual human subjects; a group of students were assembled in a computer lab to use software we developed that allowed each of them to control the color of one vertex in a graph. The participants attempted to arrive at a valid coloring of the graph in under five minutes, and they were paid for their performance. (A valid coloring is one where each node has a color that is different than the color of each of its neighbors.) We analyzed how changing the structure of the network affected the students' ability to collectively solve the graph coloring problem. We chose the graph coloring problem because it abstracts many problems that appear in computer science, logistics, and operations research. Moreover, the graphs the subjects were given to color were generated using two models from the social network theory literature, the small-world model and the preferential attachment model. Since these were human subject experiments, we designed them using the paradigm of behavioral game theory by systematically varying the parameters of the experiments and by offering monetary incentives for performance. These experiments also allowed us to compare the performance of the human subjects to distributed algorithms. This work, which originally appeared in [63], showed how arranging the same group of people in different networks affected their ability to collectively solve a problem. Thus, these experiments get at the very heart of how network topology affects the behavior of the players and the outcome of the interaction.

1.1 Economic Properties of Social Networks

Classical models of economic markets, such as the models of Fisher [42] and Arrow and Debreu [4], only consider centralized exchange where any player can exchange goods with any other player. One result of this centralized exchange is that there is no price variation. That is, if two players both have an initial endowment of the same good, say wheat, both of those players will have the same equilibrium price per unit of wheat. We wish to model situations where a buyer might have access to only certain sellers. This might arise due to geographic proximity or government regulations for example. Thus we introduce the model of bipartite exchange economies. The model consists of a bipartite network with buyers on one side and sellers on the other. Initially, each buyer starts with an infinitely divisible cash endowment; each seller starts with an endowment of an infinitely divisible abstract good, which we again call wheat. Buyers wish to trade their cash for as much wheat as possible and sellers wish to trade their wheat for as much cash as possible. Furthermore, trade can only occur between a buyer and a seller if they are attached by an edge. We define the wealth of a buyer to be the amount of wheat he gets per unit cash, and the wealth of a seller to be the amount of cash he gets per unit wheat (which is the same as the sellers price). We also define the wealth variation as the ratio of maximum to minimum wealth. We show there will be no equilibrium wealth variation among the players (i.e. all players earn the same equilibrium payoff) if and only if the underlying network contains a perfect matching. This characterizes when there will and will not be wealth variation in the graphical model, in contrast to the classical models of centralized exchange, where there is never any price variation. Furthermore, this result describes a strong relationship between network topology and equilibrium behavior.

We would also like to quantify the wealth variation as well as the wealth distribution in networks that one might see in the real world. Thus, we turn to the burgeoning field of social network theory [83, 7, 104, 103] to provide us with a generative model of networks that have similar statistical properties to those that appear in nature. One such model is called preferential attachment [95, 8, 14]. According to this model, nodes are added to the graph sequentially. When a new node is added it attaches to ν existing nodes. Furthermore, with probability α a new link connects to an existing node chosen uniformly at random, and with probability $1-\alpha$ a new link connects to an existing node with probability proportional to that existing nodes degree. This results in nodes inserted at the beginning having substantially higher degree than later nodes. In fact, the degree distribution of these graphs obeys a power-law. (We defer the formal description of this model, and the statistical properties of the graphs it generates, to Section 2.5.) When we analyze networks generated via preferential attachment as bipartite exchange economies, we prove the following theorem, which is supported by an experimental analysis, that provides tight bounds on the wealth variation in terms of statistical properties of preferential attachment graphs.

Theorem (2.6.2)**.** *In the bipartite (α, ν)-model, if $\alpha(\nu^2+1) < 1$, then the ratio of the maximum seller price to the minimum seller price scales with number of buyers n as $\Omega(n^{\frac{2-\alpha(\nu^2+1)}{1+\nu}})$. For the simplest case in which $\alpha = 0$ and $\nu = 1$, this lower bound is just $\Omega(n)$.*

In addition, we prove that the distribution of wealth in these networks also obeys a power-law, given by the following theorem.

Theorem (2.6.1)**.** *In the bipartite (α, ν)-model, the proportion of sellers with wealth greater than ω is $O(\omega^{-1/\beta})$. For example, if $\alpha = 0$ (pure preferential attachment) and $\nu = 1$, the proportion falls off as $1/\omega^2$.*

Again, we support this theoretical result with simulations. This is the first model that explains the heavy tailed distribution of wealth in the real economies, first observed by Pareto [91], in terms of purely network effects. Thus the marriage of social network theory and general equilibrium theory gives insight how the topology of social networks affects equilibrium behavior.

Another contribution of this work is more computational in nature. Our graphical model provides an efficient, local algorithm that approximates the equilibrium wealth of a seller in the following way. Let s be a seller in a bipartite exchange economy G, and let G' be the induced economy consisting of all players within distance k of s. We prove that if one computes the equilibrium wealth of s in G', and k is even, one gets a lower bound on the equilibrium price of s in G. Similarly, if one computes the equilibrium wealth of s in G', and k is odd, one gets an upper bound on the equilibrium price of s in G. For the case of preferential attachment graphs, we give an experimental analysis that shows that these two bounds converge exponentially fast in k. We also give an experimental analysis that exhibits the increase in computational speed of using this approximation method for each seller as opposed to doing one global equilibrium computation. All of these results show that the effect of topology on equilibria can be seen in the equilibrium wealth of the players, as well as in the computational aspects of finding such an equilibria.

1.2 A Network Formation Game for Bipartite Exchange Economies

In the model described in the previous section, buyers and sellers trade according to an *exogenously* defined network. We also consider a model where buyers and sellers trade according to an *endogenously* defined network. In this model, initially there is a set of buyers and a set of sellers with no edges between them. The actions of each player are to buy edges at a fixed cost of α per edge. The players trade via the same market mechanism as the previous section according to the graph that has been purchased. Each players payoff is defined as the wealth that player earns due to market trading minus α times the number of edges that player bought. We analyze the Nash equilibria of this network formation game. That is, we characterize which networks are stable in the sense that no player could increase its payoff by unilaterally deviating by buying a different set of edges.

The main result we give in this model is a complete characterization of such equilibrium graphs. We show that there are 3 families of equilibrium graphs. The first family consists of perfect matchings. In these graphs all players earn a wealth of 1 from market trading. The second family consists of disjoint monopolies where one buyer exploits many sellers or one seller exploits many buyers. As a result, we call this family of graphs Exploitation Graphs. These types of graphs occur when the cost of an edge is high enough such that an exploited buyer, for example, cannot buy an edge to an exploited seller and increase its payoff. The final family of graphs consists of many disjoint minimum spanning trees with k sellers and $k+1$ buyers (or vice versa). In these graphs the buyers and sellers earn relatively equal amounts from market trade, so we call these graphs Balanced Graphs. These types of graphs occur when the cost of an edge is relatively low. The overall result is described in following theorem.

Theorem (3.4.3)**.** *Let $NE(n, \alpha)$ be the set of all Nash equilibria graphs of the network formation game for a fixed population size n and edge cost α, and let NE be the union of $NE(n, \alpha)$ over all n and α. Then the set NE equals the union of classes Perfect Matchings, Exploitation Graphs, and Balanced Graphs.*

1.3 Networks Preserving Evolutionary Equilibria and the Power of Randomization

Next we describe the classical model of evolutionary game theory and its fundamental equilibrium concept, and then we describe how we introduce a network into this model. The classical model of evolutionary game theory [96] considers an infinite population of organisms, where each organism is assumed to be equally likely to interact with each other organism. Interaction is modeled as playing a fixed, 2-player, symmetric game. Suppose there is a $1 - \epsilon$ fraction of the population who play strategy s, and call these organisms incumbents; suppose there is an ϵ fraction of the population who play t, and call these organisms mutants. The strategy s is an evolutionarily stable strategy (ESS) if the expected fitness of incumbent is higher than that of a mutant for all $t \neq s$ and all sufficiently small ϵ [98].

Now we introduce a model of evolutionary game theory over networks. Instead of each organism being equally likely to interact with each other organism, in the graphical model each organism interacts with all of the organisms in its local neighborhood. An organisms fitness is defined to be the average fitness it would obtain if it played each one of its neighbors. This is equivalent to considering the expected fitness of an organism interacting with a randomly chosen neighbor. We say that a strategy s is stable with respect to a game F and an infinite family of graphs $\{G_n\}_{n=0}^{\infty}$ if for all families of mutants $\{M_n\}_{n=0}^{\infty}$ of linear size, all but $o(n)$ of the mutants have an incumbent neighbor of higher fitness. We will prove two complementary results of the form, s is a classical ESS of F if and only if s is an ESS with respect to F and a very broad class of graphs. We also show that if one were to require that *all* mutants have an incumbent neighbor of higher fitness then it would be impossible to have results that hold for these two classes of graphs and *any* 2-player, symmetric game. Thus we only require all but a vanishing number of mutants have an incumbent neighbor of higher fitness. In addition, we prove that this definition recovers the classical definition in the case of the network being a clique.

We now proceed to describe two complementary results in the network ESS model. First, we consider a setting where the graphs are generated via the $G_{n,p}$ model of Erdős and Rényi [13]. In this model, every pair of vertices are joined by an edge independently and with probability p (where p may depend on n). The mutant set, however, will be constructed adversarially (subject to a size constraint we will exhibit in Definition 4.3.3). For these settings, we show that for any 2-player, symmetric game, s is a classical ESS of that game, if and only if s is an ESS for random graphs, where $p = \Omega(1/n^c)$ and $0 \leq c < 1$, and any linear sized mutant family. We note that under these settings, if we let $c = 1 - \gamma$ for small $\gamma > 0$, the expected number of edges in G_n is $n^{1+\gamma}$ or larger — that is, just super-linear in the number of vertices and potentially far smaller than $O(n^2)$. The forward direction of this theorem is stated below.

Theorem (4.5.1)**.** *Let F be any 2-player, symmetric game, and suppose s is a classical ESS of F. Let the infinite graph family $G = \{G_n\}_{n=0}^{\infty}$ be drawn according to $G_{n,p}$, where $p = \Omega(1/n^c)$ and $0 \leq c < 1$. Then with probability 1, s is an ESS with respect to F and G.*

It is easy to convince oneself that once the graphs have only a linear number of edges, we are flirting with disconnectedness, and there may simply be large mutant sets that can survive in isolation due to the lack of any incumbent interactions in certain games. Thus we examine the minimum plausible edge density for random graphs. In addition to proving the above theorem, we also prove its converse, Theorem 4.5.2, in Section 4.5.1.

The second result is a kind of dual to the first. It assumes the graphs are chosen arbitrarily (subject to edge density restrictions) and the mutant sets are chosen randomly. It states that for any 2-player, symmetric game, s is a classical ESS for that game, if and only if s is an ESS for any family of graphs in which for all $v \in V$, $\deg(v) = \Omega(n^\gamma)$ (for any constant $\gamma > 0$), and a family of mutant sets that is chosen randomly (that is, in which each organism is labeled a mutant with constant probability $\epsilon > 0$). We give the formal statement of the forward direction of this theorem below.

Theorem (4.5.3)**.** *Let $G = \{G_n = (V_n, E_n)\}_{n=0}^{\infty}$ be an infinite family of graphs in which for all $v \in V_n$, $\deg(v) = \Omega(n^\gamma)$ (for any constant $\gamma > 0$). Let F be any 2-player, symmetric game, and suppose s is a classical ESS of F. Let t be any mutant strategy, and let the mutant family $M = \{M_n\}_{n=0}^{\infty}$ be chosen randomly by labeling each vertex a mutant with constant probability ϵ, where $\epsilon_t > \epsilon > 0$. Then with probability 1, s is an ESS with respect to F, G and M.*

In addition to proving the above theorem, we also prove its converse, Theorem 4.5.5, in Section 4.5.2. Thus, in this setting we again find that classical ESS are preserved subject to edge density restrictions. Again, the reason that we can characterize the evolutionarily stable incumbent strategies is that this graph topology precludes small islands of mutants that could prosper with only minimal contact with incumbents.

In both of these results the topology of the network has a direct effect on the structure of equilibria, and now we discuss what they imply about the relationship between the graphical and classical models. At an intuitive level, the first result says that if a graph is chosen randomly and the mutants are chosen adversarially, then classical ESS, and only those, are preserved. The second result says that if the graph is chosen adversarially and the mutants are chosen randomly, then classical ESS, and only those, are preserved. This shows that for the purposes of characterizing stable strategies, the random matching scheme of the classical model is equivalent to either randomizing the graph or randomizing the mutations.

1.4 An Experimental Study of the Coloring Problem on Human Subject Networks

The final main component of this work is experimental in nature. Its purpose is to investigate how the same group of people behave when organized in different networks. A group of 38 undergraduates were instructed to solve the graph coloring problem. Each person was assigned to control the color of a different node in a 38 node graph. They used a distributed software system which allowed them to change the color of their node asynchronously, and to see the other nodes changing colors in real time. Their objective was to arrive at a valid coloring (where each node has a color different than each of its neighbors) in under 300 seconds. The subjects were paid for their performance. We performed a series of experiments where we systematically varied 3 design variables. Next we describe each design variable, how we varied it, and the conclusions we draw from doing so.

We systematically varied the topology of the networks that the subjects were given to color. The subjects colored graphs chosen from the small-world [103] and preferential attachment models [95, 8], which are generative models from the social network theory literature, as well as a "leader cycle", which models corporate or military networks. The small-world family consists of graphs which have a simple cycle containing all the nodes, augmented with a variable number of chords. These graphs model nodes that have a few links to geographically close neighbors as well as a few long distance links. The graphs we chose from the small-world model were a simple cycle, a cycle with 5-chords, and a cycle with 20-chords. The subjects also colored a leader cycle, which has a more hierarchical topology. It consists of a simple cycle of 36 nodes, and two leader nodes. One leader node is connected to the 18 nodes on the cycle with even parity, and the other leader node is connected to the 18 nodes on the cycle with odd parity. The leaders are also connected to each other. The simple cycle, 5-chord cycle, 20-chord cycle, and leader cycle have a diameter of 19, 12, 7, and 3, respectively, and all of these graphs are 2-colorable. We found that as the diameter of these cycle-based graphs decreased, so did the time in which it took the subjects to color the graphs. The participants also colored 2 graphs from the preferential attachment model. These are constructed by adding nodes to the graph sequentially. Each new node attaches to an existing node with probability proportional to that existing node's degree. One preferential attachment graph was constructed by each new node attaching to 2 existing nodes, and one was constructed by each new node attaching to 3 existing nodes. The graph with 2 new links per vertex has a diameter of 5, and the graph with 3 new links per vertex has a diameter of 4. We observed that the lower diameter preferential attachment graph was colored faster than the higher diameter preferential attachment graph. One of the key findings of this experiment was that the group could color networks with low diameter much faster than networks with high diameter — showing that network topology impacted group behavior.

We also varied how much information the participants were allowed to see. In 1/3 of the experiments, the subjects could only see their color and the color of their neighbors, in 1/3 of the experiments the subjects could see their color, the color of their neighbors and their neighbors degree, and in the final 1/3 of the experiments the subjects could see the entire graph and the color of every node. We found that increasing the amount of information the subjects could see helped them color the 2-colorable, cycle based networks much faster. On the other hand, increasing the amount of information hindered their ability to color the complex preferential attachment graphs. This observation implies that in the case of human subjects, not only does the topology of the network affect behavior, but also how much information is revealed to the subjects about the topology.

The third and final design variable we varied was the incentive scheme. In 1/2 of the experiments, we used a global incentive scheme, where each subject was paid $5 if the group successfully colored the graph. In the other 1/2 of the experiments, we used an individual incentive scheme, where each subject was paid $5 if they had no color conflicts (having the same color as a neighbor) at the end of an experiment. Recall that an experiment would end if either the group found a valid coloring or 300 seconds elapsed. We found that the subjects changed color when they had no coloring conflicts over twice as many times under the global incentive scheme, than under the individual incentive scheme. A possible reason for a node to change color when it has no color conflicts is to escape a perceived local minimum in the search for a valid coloring. This shows that human subjects can change behavior quite dramatically with only subtle changes to their incentive scheme.

The motivation for these experiments is two-fold. First, the relationship between network structure and behavior is difficult to establish by observing people engaged in their natural social network. In such studies the network structure is fixed and given, thus preventing the investigation of alternatives. A different approach is to conduct controlled laboratory experiments in which network structure is deliberately varied. Second, since the pioneering "small-world" experiment [99, 78], there has been a long and fascinating literature examining the structural and navigational properties of natural social networks. Findings range from the now-familiar of "six degrees of separation" to more recent theoretical explanations of the heuristics people might employ to exploit social network structure [28, 105, 68, 69]. This line of investigation can be summarized in computer science terminology: Using relatively local information, distributed human organizations can collectively compute good approximations to the all-pairs shortest paths problem. Given the volume and visibility of this research, it is perhaps surprising that there is little work on its natural generalization — namely, what *other* types of distributed optimization problems can humans networks solve? These experiments begin to answer this question.

1.5 Related Work

In this section we discuss previous research that is relevant to the general topic of relating network topology to equilibrium structure. We defer the discussion of work that is related to only one of the specific models proposed in this thesis to the chapter dedicated to that model. First, we outline the previous work that has inspired this general line of questioning. Then, we describe other work that exhibits a relationship between network topology and equilibrium behavior. Finally, we briefly mention algorithmic research that leverages network structure to compute equilibria.

Much of the work in this thesis is inspired by the notion of a graphical game introduced in [60] and the notion of a graphical economy introduced in [55]. We discuss these two concepts in turn. We give the formal definition of a graphical game below (taken from [59]), but first we define some necessary notation. In a graphical game player i is identified with vertex i in an undirected graph $G = (V, E)$. Let $N(i)$ denote the neighborhood of i, that is $N(i) = \{j | (i, j) \in E\} \cup \{i\}$. Let $\vec{a}$ denote a vector describing the actions of all players, and let $\vec{a}^i$ denote the projection of $\vec{a}$ onto the players of $N(i)$.

Definition 1.5.1. *A* graphical game *is a pair* $(G, \mathcal{M})$ *where* G *is an undirected graph over the vertices* $\{1, \ldots, n\}$*, and* $\mathcal{M}$ *is a set of* n *local game matrices. For any joint action* $\vec{a}$*, the local game matrix* $M_i \in \mathcal{M}$ *specifies the payoff* $M_i(\vec{a}^i)$ *for player* i*, which depends only on the actions taken by the players in* $N(i)$*.*

The contributions of this concept are myriad. First, since the influence on each players payoff is explicity described in the graph G, one can now ask questions such as: how does the topology of the graph G affect the structure of the equilibrium. This is the central question of this thesis. Second, this definition allows for the compact representation of a normal form game. To represent a game with n players, where each player has 2 actions in tabular form, would require a matrix of size $n2^n$. In the case where the payoff of each player depends only a maximum of d other players, the graphical game representation only requires space exponential in d which may be considerably smaller than n. Third, algorithms can leverage the expressiveness of the graph G to compute equilibria of graphical games with restricted topologies in cases where computing equilibria of the general game is intractable.

In [55] the authors introduced the model of graphical economies which imposes a graphical framework onto market-based games. In this model each player is identified with a node in an undirected graph, and players can only trade with players they are connected to. Each player has a continuous, monotone, and quasi-concave utility function that describes their preferences for different bundles of goods. Each player starts off with an initial bundle of goods, and there is a price vector that describes the prices of the goods sold by each player. Players can trade their goods with others in their neighborhood according to this price vector and their utility functions in order to get a bundle which gives them higher utility. An equilibrium is reached if each players only end up with a bundle that makes them optimally happy, given their budget and the local prices, and the supply of each good in a neighborhood equals its demand in that neighborhood. The authors give very general conditions as to when such an equilibrium exists, as well as algorithms for computing such equilibria. More importantly, from the perspective of this work, they show that there can be equilibrium price variation in this model. That is, at equilibrium two players could command different prices for the same good strictly due to the differing positions these two players occupy in the graph. The authors do not quantify how much price variation there could be; they merely show that it could exist.

Thus, graphical games provide a formalism in which one can begin to explore the relationship between network topology and equilibrium structure, and graphical economies show that there can indeed be a causal relationship between the two. This thesis seeks to quantify and describe that relationship. Next we survey other results that relate network topology to structure of equilibria.

The authors of [74] introduced the model of interdependent security (IDS) games. We describe this model using the example of airline baggage security. When a bag is first checked into an airline it is screened for explosive devices. But, when bags are transferred from one airline to another, the receiving airline usually does not perform any extra screening. Thus each airline has a direct risk, the risk of having an explosive device checked directly in to that airline, as well as an indirect risk, the risk of having an explosive device transferred from another airline. These two sources of risk, direct, which an airline can mitigate, and indirect, which an airline has largely no control over, are the two key notions behind IDS games. In [61] the authors hypothetically assume there is a new technology that could determine with absolute certainty whether or not a bag has an explosive device in it or not. Using real airline travel data they estimate the risk each airline would face from transfers. With these estimates, they experimentally show that all airlines would converge to not investing in the new security. The authors also show, that if the 3 largest airlines were forced to invest in the security, all airlines would converge to investing in the security. Thus, they show that the equilibrium behavior of this network based game exabits a tipping phenomenon based on which airlines invest. This is another result which relates the topology of the graph to structure of equilibria. However, in this thesis we seek to relate structure to equilibrium in more than just one experimentally derived graph. We wish to give theoretical results that relate topology to equilibrium in large families of graphs.

In [66] the authors consider a game theoretic model with n players, where players can communicate with other players according to a random graph G generated via the $G_{n,p}$ model of Erdős and Rényi [13] . They say an action profile can be blocked by a coalition of players C if the players in the coalition can deviate from the profile and all earn a higher payoff for doing so. They analyze one setting where a coalition may form if all its members form a clique in the communication graph G. They show that if $p = \Omega(1/\log n)$, then any non-Pareto optimal allocation will almost surely be blocked. They also analyze a setting where a coalition may form if any 2 members have a neighbor in common that is also in the coalition. Here they show that if $p = \Omega(1/\sqrt{n})$, then any non-Pareto optimal allocation will almost surely be blocked. This model differs from those we consider because it allows for situations where any players action can affect the payoff of any other player, whereas we consider models where the payoff of one player can only be affected by players in its local neighborhood. In spite of this, the authors prove that the graphical restriction on which coalitions can be formed influences which types of equilibria could result.

The work on cooperative transferable utility games lies in a similar vain as the work described in the previous paragraph. Transferable utility games consist of n players and a characteristic function $\nu : 2^n \rightarrow \Re$. Each player can choose to cooperate or not. Those that do form a coalition S, and $\nu(S)$ is the total payoff that the coalition receives. As in the work mentioned previously by [66], communication is often restricted to occur over a fixed undirected graph. Players can only choose to cooperate if they are connected via this communication graph. The work of Borm et al. [15] surveys the literature that relates the topology of this communication graph to the payoff of the coalition. This is another example of a class of games where each players action can affect each other players payoff, but the structure of the coalition is governed by a graph. In this work, we analyze models where each players action can only affect the payoffs of the others in their neighborhood.

In [77] the authors describe an alternate way of interpreting Nash equilibrium. In this alternate formulation each player corresponds to a population of agents, where each agent always plays one pure strategy of the player. The relative frequency of the agents in a population corresponds to the mixed strategy of that player. The game is played by choosing an agent uniformly at random from each population to play. An equilibrium of this game corresponds to a Nash equilibrium of the underlying game, which has 1 player per population choosing a mixed strategy corresponding to the frequency agents. The authors then consider what happens if the agents are not necessarily chosen uniformly at random from each population. Let μ denote a distribution over the populations that governs which agents meet. This defines a "local-interaction" game because agent 1 of population 1 might meet agent 1 of population 2 but never meet agent 2 of population 2. The authors show that any Nash equilibrium of the local-interaction game corresponds to a correlated equilibrium of the underlying game, and *vice versa*. They show that pattern of local interactions defined by μ can be encoded in the signal that the players receive in the correlated equilibrium. This is a different type of local interaction than we consider in this work, however, the authors do relate the topology of the local interaction to the signals given to players which shapes the structure of the equilibria.

Finally, we briefly describe algorithms that make use of the topology of graphical games to compute equilibria. For a much more extensive introduction into this area we refer the reader to [59]. In [60] the authors give two algorithms that compute Nash equilibria for graphical games with a tree topology. The first runs in polynomial time and outputs a compact representation of an approximation of every Nash equilibrium, and the second runs in exponential time and outputs all Nash equilibrium exactly.

This was later generalized in [87] to general graphs with cycles, although the authors were only able to give an experimental analysis of the running time. In [33] the authors give a polynomial time algorithm for computing Nash equilibria in graphs where the maximum degree is 2. It is based on the TreeNash algorithm of [60]. On the lower bound side, [46, 26] show that finding the Nash equilibria of graphical games where maximum degree node has degree at least 3 is PPAD-complete and thus is likely to be intractable. In [33] the authors extend this result to show there exists a constant k such that it is PPAD-complete to find a Nash equilibria of a graphical game where the graph has pathwidth k and every node has degree at most 3. This shows that the computability of graphical games is very sensitive to the topology of underlying graphs.

The structure of a graphical game can also be exploited to compute correlated equilibria. In [54] the authors show if G is the underlying graph of a graphical game, then there exists a Markov network that is almost identical to G, that represents every correlated equilibria of a graphical game. They then go on to exploit this relationship to give a polynomial time algorithm for computing correlated equilibria in graphical games where the underlying graph is a tree. Subsequently, [90, 89] gave algorithms that can not only compute correlated equilibria in a more general class of games with a compact representation, but also allow for optimization over which correlated equilibria is computed.

Chapter 2

Economic Properties of Social Networks

2.1 Introduction

There is a long history of research in economics on mathematical models for *centralized* exchange markets, and the existence and properties of their equilibria. The work of [4], who established equilibrium existence in a very general commodities exchange model, was certainly one of the high points of this continuing line of inquiry. The origins of the field go back at least to [42]. Almost all of the models studied in this line of research assume that each player is able to trade with each other player.

While there has been relatively recent interest in network models for interaction in economics (see [51] for a good review), it was only quite recently that a network or graph-theoretic model that generalizes the classical Arrow-Debreu and Fisher models was introduced [55]. In this model, the edges in a network over individual consumers (for example) represent those pairs of consumers that can engage in direct trade. As such, the model captures the many real-world settings that can give rise to limitations on the trading partners of individuals (regulatory restrictions, social connections, embargoes, and so on). In addition, variations in the equilibrium payoff or wealth of a player can arise due to the topology of the network: certain individuals may be relatively favored or cursed by their position in the graph.

In a parallel development over the last decade or so, there has been an explosion of interest in what is broadly called *social network theory* — the study of apparently "universal" properties of natural networks (such as small diameter, local clustering of edges, and heavy-tailed distribution of degree), and statistical generative models that explain such properties. When viewed as economic networks, the assumptions of individual rationality in these works are usually either non-existent, or quite weak, compared to the Arrow-Debreu or Fisher models.

In this chapter we will first formally define our network-based economic exchange model, and then we will analyze how the topology of the underlying network affects the structure of the equilibria. First, we completely characterize when there will and will not be equilibrium wealth variation for any graph. Second, we describe a local approximation method that allows one to compute global approximate equilibrium wealth using only local regions of the network. Then we will analyze our model in the modern light of social network theory. We are particularly interested in the interaction between the statistical structure of the underlying network and the variation in wealth at equilibrium. We quantify the amount of wealth variation that that certain generative models (such as the preferential attachment model of network formation [95, 8] are capable of. This will give the first graph-theoretic explanation of the heavy-tailed distribution of wealth in the real world first observed by Pareto [91]. We also show that in the preferential attachment model, prices computed from only local regions of a network yield strikingly good estimates of the global prices.

2.2 Related Work

The inspiration for this work comes from [55]. There the authors introduced the notion of a graphical economy which combines networks with game theoretic models of markets. The authors of that work give very general conditions for the existence of equilibria and algorithms for computing such equilibria. (We refer the reader to Section 1.5 for more details about this work.) But, the authors did not tackle the question of how the topology of the underlying network affects the equilibrium payoffs of the players. This is the main question this chapter addresses.

There are 2 specific works from the economics literature that are most closely related to this chapter. First, Corominas-Bosch [25] recently analyzed a similar model to the one we consider in this chapter. Her model consists of a bipartite network with buyers on one side and sellers on the other. Each buyer starts off with 1 unit of cash which is divisible, and each seller starts off with 1 unit of the same indivisible good. Trade occurs via a bargaining mechanism (as opposed to the market mechanism we consider) that proceeds in rounds. During the odd numbered rounds each seller proposes a price to the buyers in its neighborhood, and the buyers choose whether or not to accept the price of one seller they are connected to. Similarly, during the even numbered rounds each buyer proposes a price to the sellers in its neighborhood, and the sellers choose whether to accept the price of one buyer they are connected to. When a price between a buyer and a seller is accepted, the two trade and are removed from the graph, the process then continues. Due to this bargaining mechanism, the equilibrium graphs can be decomposed into 3 types of subgraphs (those where the number of buyers is greater than, equal to, and less than the number of sellers). Moreover, each type of subgraph has a very simple and precise equilibrium price structure. For example in the case of a subgraph with a greater number of sellers than buyers, all sellers earn an equilibrium price of 0 and all the buyers get the good for free. While, this result does relate network topology to structure of the equilibrium, in this chapter we will show that the range in prices in the market mechanism we consider will exhibit a much richer structure. Furthermore, we will quantify that rich structure in terms of topological and statistical properties of the network.

Second, Kranton and Minehart [72] also analyze a model which consists of a bipartite network with buyers on one side and sellers on the other. Each seller has an initial endowment of 1 indivisible unit of a good. Each buyer demands 1 unit of that good and has a valuation for it. The key structure that underlies the results in this work is the notion of an opportunity path. For any feasible allocation A, an opportunity path starts at a buyer and ends at some other buyer and consists edges that alternate between edges over which trade did occur according to A, and edges over which trade did not occur according to A. The authors show that if a buyer buys a good, the price it pays can be at most the price paid by buyers along the opportunity path. The authors then use this result to give bounds on the equilibrium prices in terms of the valuations of the nodes along the opportunity path. In this chapter, however, we prove a slightly different type of result. We will prove bounds that quantify the equilibrium payoffs of the players in purely terms of the node and edge structure of the graph (and not of the valuations or the utility functions of the players).

2.3 Bipartite Exchange Economies

A bipartite exchange economy consists of a bipartite graph $G = (V = B \cup S, E)$, where nodes on one side of the bipartition represent buyers (B), and nodes on the other side of the bipartition represent sellers (S), and all edges in E are between B and S. There are two abstract commodities that, without loss of generality, we shall call *cash* and *wheat*. Buyer i has an infinitely divisible endowment of 1 unit of cash to trade for wheat; seller j has an infinitely divisible endowment of 1 unit of wheat to trade for cash. Buyers have utility x for x units of wheat and 0 utility for cash; similarly, sellers have utility x for x units of cash and 0 utility for wheat[1]. The

[1] The exact form of these functions is irrelevant as long as each party has non-zero and increasing utility only for the "other" good.

semantics of the graph are as follows: buyer i can trade with seller j if and only if there is an edge between i and j.

Before describing the standard notion of equilibrium for this model, we note that it is a significant and deliberate specialization of the model first considered in [55], which among other features permitted varying initial endowments and utility functions, as well as an arbitrary number of commodities. Our interests here are in the structures that arise purely from "network effects", as opposed to those arising from imbalances in supply and demand, variations in consumer utilities, and so on. Thus, we make the endowments and utility functions of the players identical so we can ascribe any variation in the equilibria solely to variation in the underlying graph.

We now describe our notion of exchange equilibrium for a bipartite exchange economy. Let ω_j^s denote the exchange rate (or price), in terms of cash per unit wheat, that seller j is offering. Similarly, let ω_i^b denote the exchange rate, in terms of wheat per unit cash, that buyer i is offering. Let x_{ij} denote the amount of seller j's wheat that buyer i consumes. A set of exchange rates, $\{\omega_i^b\}$ and $\{\omega_j^s\}$, and consumption plans, $\{x_{ij}\}$, constitutes an *exchange equilibrium for* G if the following two conditions hold [56]:

1. The market *clears*, *i.e.* supply equals demand. More formally, for each seller j, $\sum_{i \in N(s_j)} x_{ij} = 1$ where $N(i) = \{j | (i,j) \in E\}$. The value of 1 on the right hand side represents j's endowment.

2. For each buyer i, their consumption plan $\{x_{ij}\}_j$ is optimal. By this we mean that according to the consumption plan, buyers only buy from the sellers *in their neighborhood* offering the *best* exchange rate. That is, $x_{ij} > 0$ if and only if $\omega_j^s = \min_{s_k \in N(b_i)} \omega_k^s$.

We note that the role of buyers and sellers in a bipartite exchange economy is completely symmetric. Given buyer i's exchange rate, ω_i^b, one can determine how much of buyer i's cash seller j consumes. Thus, one could equivalently define Item 1 above from the point of view of the buyers and Item 2 above from the point of view of the sellers. In this model, an exchange equilibrium for G always exists if each seller has at least one neighboring buyer [32]. Furthermore, the equilibrium exchange rates are unique, and at equilibrium, if $x_{ij} > 0$ then $\omega_j^s = 1/\omega_i^b$.

Since each seller starts off with 1 unit of wheat and his exchange rate[2] is in terms of cash per unit wheat, at exchange equilibrium each seller will earn exactly his exchange rate in dollars. Thus we will also call each sellers exchange rate ω_j^s her *wealth*. Similarly, at exchange equilibrium each buyer will earn exactly his exchange rate in wheat, so we call the buyers exchange rate ω_i^b her wealth as well. We say there is *no wealth variation* at exchange equilibrium of a bipartite exchange economy when the wealth of all of the sellers are equal and the wealth of all of the buyers are equal. We say there is wealth variation at exchange equilibrium if some buyers earn a different amount of wealth than other buyers and/or some sellers earn a different amount of wealth than other sellers. In a bipartite exchange economy where the number of buyers and sellers are equal, at exchange equilibrium the average wealth will be 1, so some player has wealth less than 1 if and only if some other player has wealth greater than 1.

2.4 Results: General Networks

In this section we will describe the results that relate the topology of an *arbitrary* graph or network to the structure of exchange equilibrium. First, we will describe our results that relate topology to wealth variation. Second, we will show how one can use the graphical nature of our model to compute a local approximation of global equilibrium wealth. We will use these results later in Section 2.6 where we analyze the relationship between topology and equilibrium specifically for social networks.

[2]We will also call a sellers exchange rate her price.

2.4.1 Wealth Variation

In this subsection we prove the following theorem that completely characterizes when there is wealth variation in a bipartite exchange economy.

Theorem 2.4.1. *There is no wealth variation in a bipartite exchange economy if and only if there is a perfect matching in the underlying graph.*

From the statement of this theorem one can immediately see there is a strong connection between network topology and equilibrium structure in a bipartite exchange economy. We use the following two lemmas to prove this theorem. In doing so, we use the following notation. For a graph $G = (V, E)$, and $X \subseteq V$ we let $N(X) = \{y \in V | (x, y) \in E \text{ for some } x \in X\}$.

Lemma 2.4.1. *If for all subsets of sellers S, $|N(S)| \geq |S|$, then there is* no *price variation,* i.e. all *seller prices are equal to* 1.

Proof. Let us proceed with a proof by contradiction by assuming that all seller prices are not equal to 1. This implies that there must exist some sellers with equilibrium price less than 1 (if all equilibrium prices were greater than 1 than the market would not clear). Let S be the set of all sellers with price less than 1. All buyers in $N(S)$ will buy only from sellers in S since the prices outside of S are strictly greater (they are 1 or larger). Hence, the clearance condition implies that all the $|N(S)|$ buyers will spend *all* of their money within S. By assumption $|N(S)| \geq |S|$, thus the pigeonhole principle shows there must be at least one seller in S who earns at least \$1. This contradicts the definition of S. $\square$

The following lemma provides a converse to Lemma 2.4.1.

Lemma 2.4.2. *If there exists a subset of sellers S, such that $|N(S)| < |S|$, then there is price variation,* i.e. not all *seller prices will be equal.*

Proof. Due to the market clearing condition, each of sellers in S will sell all of their 1 unit of good. Since the sellers of S are only connected to those buyers in $N(S)$, they will receive at most $N(S)$ dollars which will be allocated to the $|S|$ sellers in some manner. Thus, there must be at least one element of S that gets at most $|N(S)|/|S| < 1$ dollars. Since there is a seller that gets strictly less than 1 dollar, by the market clearing condition, there must be some other seller that gets strictly greater than 1 dollar. Thus, there is price variation. $\square$

Combining Lemmas 2.4.1 and 2.4.2 we get the following theorem.

Lemma 2.4.3. *A necessary and sufficient condition for there to be* no *seller price variation,* i.e. *for* all *seller prices to be equal to* 1*, is that for all subsets of sellers* S*,* $|N(S)| \geq |S|$.

This can be viewed as an extremely weak version of standard *expansion* properties well-studied in graph theory and theoretical computer science — rather than demanding that neighbor sets be strictly larger, we simply ask that they not be smaller. A symmetric argument can be made to show that there will be no wealth variation among the buyers, if for all subsets of buyers B, $|N(B)| \geq |B|$. One can further show that for large n, the probability that a random graph (for any edge probability $p > 0$) obeys this weak expansion property approaches 1. In other words, in the Erdős-Rényi model model, there is no variation in price — a stark contrast to the preferential attachment results we will show in Section 2.6.2. Next we state Hall's Theorem taken from [24].

Theorem 2.4.2. *There exists a perfect matching in a bipartite graph* $G = (L \cup R, E)$ *if and only if* $|A| \leq |N(A)|$ *for every subset* $A \subseteq L$.

Combining Lemma 2.4.3 and Theorem 2.4.2 we get Theorem 2.4.1.

2.4.2 Local Approximation Method

This section describes a method for approximating the wealth a player would earn at exchange equilibrium which only uses the local neighborhood of that player in the bipartite exchange economy. We will present a rather intuitive "monotonicity" lemma, which states that if the supply of goods in a graphical economy is decreased, or the cash endowments are increased, then the equilibrium prices increase or remain the same. The vehicle for proving this lemma is the algorithm of Devanur et al. [27]. This algorithm computes the market clearing equilibrium of a graphical Fisher economy which is a more general type of economy than a bipartite exchange economy. Next we will introduce the graphical Fisher economy, and see how it relates to the bipartite exchange economy. Then we will state and prove the monotonicity lemma. Finally, we will state and prove the frontier bound which will exhibit our approximation method.

Market Economies on Networks

At a high level the graphical Fisher model is more general than a bipartite exchange economy for the following reasons.

1. A bipartite exchange economy requires an equal number of buyers and sellers, whereas the graphical Fisher model allows for an unequal number of buyers and sellers.

2. A bipartite exchange economy allows for only two goods in the economy, cash and wheat. The graphical Fisher model allows for multiple goods.

3. In a bipartite exchange economy the initial endowments for both buyers and sellers are uniform. In the graphical Fisher model, the players can have variable endowments.

4. In a bipartite exchange economy buyers are assumed to have utility only for wheat. In the graphical Fisher model buyers are assumed to have linear utility functions over the various goods the sellers sell.

More formally, we assume without loss of generality that each seller sells only one of the available goods. That is, seller j has an initial endowment of g_j units of an infinitely divisible good j. Buyer i has an initial endowment of e_i units of an abstract, infinitely divisible good which we call cash. Furthermore, each buyer is assumed to have a utility function that is is *linear* in the amount of goods consumed. Let $u_{ij} \geq 0$ denote the utility derived by i on obtaining a single unit of good j. If i consumes x_{ij} amount of good j, then the utility i derives is $\sum_j u_{ij}x_{ij}$.

Next, we give the definition of an equilibrium for the graphical Fisher model. A set of *prices* $\{p_j\}$ and *consumption plans* $\{x_{ij}\}$ constitutes a *market clearing equilibrium* if the following two conditions hold:

1. The market *clears*, *i.e.* supply equals demand. More formally, for each seller j, $\sum_{i \in N(s_j)} x_{ij} = g_j$ where $N(i) = \{j|(i,j) \in E\}$.

2. For each consumer i, their consumption plan $\{x_{ij}\}_j$ is optimal. By this we mean that the consumption plan maximizes the linear utility function of i, subject to the constraints that buyers only buy from sellers in their neighborhood, and the total cost of the goods purchased by i is not more than the endowment e_i.

A market clearing equilibrium always exists if each seller j has a buyer in its neighborhood who has a positive initial endowment and derives nonzero utility for good j — that is, $e_i > 0$ and $u_{ij} > 0$ for some $i \in N(j)$. Furthermore, the equilibrium prices are unique [32].

With the definition of a graphical Fisher economy in hand, we can now state and prove the Monotonicity Lemma. Intuitively, his result shows that in a graphical Fisher economy, if the supply of goods in the economy increases, the equilibrium prices decrease, and if the amount of money in the economy increases the equilibrium prices increase. The statement of the lemma requires that the endowments be strictly greater than 0, this is to ensure that equilibria exist and are unique via the result of Eisenburg and Gale [32].

Lemma 2.4.4. *(Monotonicity) Let E and E' be two graphical Fisher economies with the same number of buyers and sellers and identical linear utility functions. If for* *all goods j and buyers i, we have $0 < g'_j \leq g_j$ and $e'_i \geq e_i > 0$ (where the primes denote quantities for economy E'), then the market clearing equilibrium prices satisfy $p'_j \geq p_j$ for all j.*

Proof. To prove this, we use properties of a recent algorithm for computing market clearing equilibria in the graphical Fisher model [27], which we now describe. The algorithm is an iterative scheme in which prices $\{\tilde{p_j}\}$ are increased at every iteration, until a market clearing equilibrium is reached. Importantly, the algorithm can be initialized to any prices which obey the following property, which is referred to as the "Invariant" in [27][3]. Define the "bang per buck" for buyer i consuming good j at price $\tilde{p}_j$ as $u_{ij}/\tilde{p}_j$. We say that the Invariant holds at prices $\{\tilde{p_j}\}$ if the buyers have enough cash to purchase all the goods in the market, while only purchasing goods which maximize their bang per buck (though the buyers may have left over cash after this purchase). It is only optimal for buyer i to purchases those goods which have maximal bang per buck.

Since the algorithm only increases the prices, to prove the lemma, it suffices to show that we can initialize this algorithm, when given input E', to the equilibrium prices, $\{p_j\}$, of E. To show that such an initialization is sound, we only need to show that the prices $\{p_j\}$ satisfy the Invariant with respect to E'. To show this, first note that since these prices are a market clearing equilibrium in E, then the buyers can use their cash endowments of $\{e_j\}$ to clear an amount of goods $\{g_j\}$, while only purchasing goods which maximize their bang per buck. Since the utility functions in E and E' are identical, each good that maximizes buyer i's bang per buck at equilibrium with respect to E, also maximizes buyer i's bang per buck upon initialization of the algorithm on E'. Thus, the buyers in E' can use larger cash endowments of $\{e'_j\}$ to clear a smaller amount of goods $\{g'_j\}$, while only purchasing goods which maximize their bang per buck . $\square$

[3]Devanur et al. [27] choose a particular initialization, but the algorithm is correct for any choice of initial prices which obey the Invariant.

Next, we present a rather intuitive "frontier" bound, which implies a method in which we can find upper and lower bounds on the equilibrium prices in a graphical Fisher economy using only *local* computations. Some definitions are required to state the theorem. Let $G = (V = B \cup S, E)$ be a graphical Fisher economy. First, note that any subset V' of buyers and sellers defines a natural *induced economy*, where the induced graph G' consists of all edges between buyers and sellers in V' that are also in G. We say that G' has a *buyer (respectively, seller) frontier* if on every (simple) path in G from a node in V' to a node outside of V', the last node in V' on this path is a buyer (respectively, seller).

Theorem 2.4.3. *(Frontier Bound) Let $G = (V = B \cup S, E)$ be a graphical Fisher economy. If V' has an induced subgraph G' with a seller (respectively, buyer) frontier, then the market clearing equilibrium price of any good j in the induced economy on V' is a lower bound (respectively, upper bound) on the market clearing equilibrium price of j in G.*

Proof. We prove the lower bound for the seller frontier case. The upper bound follows by a symmetric argument. Let S' be the set of sellers on the frontier of G'. Let B' be the set of buyers "on the other side" of the frontier, that is $B' = N(S') \cap (V \setminus V')$. Next, we will modify the initial cash endowments of the buyers in B'. Fix any $\epsilon > 0$. For each buyer $i \in B'$ set e_i to ϵ/n, and let E denote this new, modified graphical Fisher economy. By the Monotonicity Lemma (Lemma 2.4.4), the equilibrium wealth[4] of each seller s in E, denoted ω_s^E, is a lower bound on the equilibrium wealth of s in G, denoted ω_s^G.

Consider a seller $s \in G'$. The equilibrium wealth of s in E is composed of cash from buyers in G', and cash from buyers in B'. Let $\omega_s^{G'}$ denote the total amount of cash seller s gets from buyers in G'. Similarly, let $\omega_s^{B'}$ denote the total amount of cash seller s gets from buyers in B'. Thus, $\omega_s^{G'} + \omega_s^{B'} = \omega_s^E \leq \omega_s^G$. Since s can trade

[4]Recall that since seller s has an initial endowment of 1 unit of wheat, it's equilibrium wealth is the same as it's equilibrium price.

with at most n buyers in B', $w_s^{B'} \leq n\epsilon/n = \epsilon$. Since ϵ is chosen arbitrarily $\omega_s^{B'} \leq 0$. Thus $\omega_s^{G'} \leq \omega_s^{G}$. □

This theorem implies a simple price upper bound: the price commanded by any seller j is bounded by its degree d. We will see that degree alone is a fairly poor approximation of an individual sellers equilibrium price in Section 2.6.3. We will also see how fast the upper and lower bounds converge to each other.

2.5 Generative Models for Social Networks

The simplest generative model for the bipartite graph G might be the *random graph*, in which each edge between a buyer i and a seller j is included independently with probability p. This is simply the bipartite version of the classical Erdős-Rényi model [13].

Many researchers have sought more realistic models of social network formation, in order to explain observed phenomena such as low diameter and heavy-tailed degree distributions (see for example [8, 103, 69, 70, 20]). We now describe a slight variant of the *preferential attachment* model [79] for the case of a bipartite graph. We start with a graph in which one buyer is connected to one seller. At each *time step*, we add one buyer and one seller as follows. With probability α, the buyer is connected to a seller in the existing graph uniformly at random; and with probability $1 - \alpha$, the buyer is connected to a seller chosen *in proportion to the degree* of the seller (preferential attachment). Simultaneously, a seller is attached in a symmetric manner: with probability α the seller is connected to a buyer chosen uniformly at random, and with probability $1 - \alpha$ the seller is connected under preferential attachment. The parameter α in this model thus allows us to move between a pure preferential attachment model ($\alpha = 0$), and a model closer to classical random graph theory ($\alpha = 1$), in which new parties are connected to random extant parties[5].

Note that the above model always produces trees, since the degree of a new party is always 1 upon its introduction to the graph. We thus will also consider a variant of this model in which at each time step, a new seller is still attached to exactly one extant buyer, while each new buyer is connected to $\nu > 1$ extant sellers. The procedure for edge selection is as outlined above, with the modification that the ν new edges of the buyer are added without replacement — meaning that we resample so that each buyer gets attached to exactly ν distinct sellers. The main purpose of the introduction of ν is to have a model capable of generating highly cyclical (non-tree) networks, while having just a single parameter that can "tune" the asymmetry between the (number of) opportunities for buyers and sellers. There are also economic motivations: it is natural to imagine that new sellers of the good arise only upon obtaining their first customer, but that new buyers arrive already aware of several alternative sellers.

In the sequel, we shall refer to the generative model just described as the *bipartite* (α, ν)*-model*. We will use n to denote the number of buyers and the number of sellers, so the network has $2n$ vertices. In the following section, we provide results on the statistics of these networks. In Section 2.6 we will use these results to relate the statistical properties of these graphs to the structure of the equilibria.

2.5.1 Statistical Properties of the Bipartite (α, ν)-Model

In this subsection we will analyze properties of the degree distribution of the sellers in the bipartite (α, ν)-model. We will later use these properties along with the Frontier Bound (Theorem 2.4.3) to bound the equilibrium wealth distribution and the amount of wealth variation. Let $Y(j, n)$ denote the degree of the j^{th} seller (in

[5] We note that $\alpha = 1$ still does not exactly produce the Erdős-Rényi model due to the incremental nature of the network generation: early buyers and sellers are still more likely to have higher degree.

order of arrival) at time n, and let $y_{j,n} := E[Y(j,n)]$. Let γ and β be defined as follows,

$$\gamma := \frac{\alpha\nu}{\beta} \quad , \quad \beta := (1-\alpha)\frac{\nu}{1+\nu+o(1)} \, .$$

Lemma 2.5.1 will show that the $o(1)$ term in the denominator of β is an artifact of each new node attaching to ν existing nodes chosen via sampling without replacement. The following two lemmas, when combined, show that for any value of j, $\lim_{n\to\infty} \frac{Y(j,n)}{\gamma\cdot(n/j)^\beta} = 1$.

Lemma 2.5.1. *For any vale of j, as $n \to \infty$, the sequence of $(0,\infty)$-valued random variables, $\left\{ \frac{Y(j,n)}{\gamma \cdot (n/j)^\beta} \right\}$ have means $1 + o(1)$.*

Proof. We will show that

$$y_{j,n} := E[Y(j,n)] = (\gamma + O(1/j))(n/j)^\beta \, .$$

Fix j and set $Y(n) = Y(j,n)$. The total number of edges after time n is $(1+\nu)n - \nu(\nu+1)/2$. From time n to $n+1$, one of the ν additional edges is attached to seller j with probability

$$\begin{aligned} &\Pr(\text{1st edge attaches to } j) \;+ \\ &\qquad \sum_{i=2}^{\nu} \Pr(i\text{th edge attaches to } j \mid \text{edges } 1,\ldots,i-1 \text{ did not attach to } j) \end{aligned}$$

$$= \sum_{i=1}^{\nu} \frac{(1-\alpha)Y(n)}{(1+\nu)(n-\nu/2) - \sum_{k=1}^{i-1} d_k^{n+1}} + \sum_{i=1}^{\nu} \frac{\alpha}{n-(i-1)} \, . \tag{2.1}$$

Here d_k^{n+1} denotes the degree of the existing node that the kth edge attaches to during the $n+1$st timestep. Recall that in this model new edges are attached to existing nodes chosen via sampling without replacement. The model of Flaxman et al. [43] analyzes the maximum degree node in an identical model except using sampling with replacement. Since sampling with replacement can only increase the number of edges attached to the maximum degree node at each time step, we can use Theorem 1.1 of Flaxman et al. [43] to show that the maximum degree in the bipartite(α, ν) model is $o(n)$. Thus Equation 2.1 becomes

$$(1-\alpha)\frac{\nu Y(n)}{(1+\nu+o(1))(n-\nu/2)} + \alpha\frac{\nu}{n-O(1)} . \tag{2.2}$$

Letting y_n denote $E[Y(n)]$ results in,

$$y_{n+1} = y_n\left(1+(1-\alpha)\frac{\nu}{(1+\nu+o(1))(n-\nu/2)}\right) + \alpha\frac{\nu}{n-O(1)} .$$

Consequently, we have the formula

$$\begin{aligned} y_n &= y_j \prod_{k=j}^{n-1}\left(1+\frac{\beta}{k-\nu/2}\right) \\ &\quad + \sum_{r=j}^{n-1}\alpha\frac{\nu}{r-O(1)}\prod_{k=r+1}^{n-1}\left(1+\frac{\beta}{k-\nu/2}\right) . \end{aligned} \tag{2.3}$$

Next we will use the following estimate in Equation 2.3.

$$\prod_{k=j}^{n-1}\left(1+\frac{\beta}{k-\nu/2}\right) = \exp\left(\sum_{k=j-\nu/2}^{n-1-\nu/2}\log(1+\beta/k)\right) \tag{2.4}$$

$$= \exp\left(\sum_{k=j-\nu/2}^{n-1-\nu/2}\beta/k + O(1/k^2)\right) \tag{2.5}$$

$$= \exp\left(\beta\log n - \beta\log j + O(1/j)\right) \tag{2.6}$$

$$= (n/j)^{\beta}(1+O(1/j)) \tag{2.7}$$

Equation 2.5 comes from the Taylor series expansion of $\log(1+x)$. Equation 2.6 uses the estimate $\sum_{i=1}^{n} 1/i = \log n + \gamma + O(1/n)$, where γ is Euler's constant, along with applying the Euler-Maclaurin formula to the $O(1/k^2)$ term. Equation 2.7 uses the Taylor series expansion of e^x. Next, we use the estimate in Equation 2.7 in Equation 2.3. Then we apply the Euler-Maclaurin formula to the summation to complete the lemma.

$$\begin{aligned} y_n &= (1+O(j^{-1}))\left(y_j\frac{n^{\beta}}{j^{\beta}} + \alpha\nu n^{\beta}\sum_{r=j}^{n-1} r^{-\beta-1}\right) \\ &= (1+O(j^{-1}))\left(y_j+\frac{\alpha\nu}{\beta}\right)\frac{n^{\beta}}{j^{\beta}} . \end{aligned} \tag{2.8}$$

□

To state the next lemma we will need the following definition.

Definition 2.5.1. *A sequence of real valued random variables* $\{X_n\}_{n=1}^{\infty}$ *is tight if for all* $\epsilon > 0$ *there exists an* $M_\epsilon > 0$ *such that for all* n, $\Pr(|X_n| \geq M_\epsilon) < \epsilon$.

Lemma 2.5.2. *For any value of* j*, as* $n \to \infty$*, the sequence* $\left\{ \frac{Y(j,n)}{\gamma \cdot (n/j)^\beta} \right\}$ *is tight as a sequence of* $(0, \infty)$*-valued random variables.*

Proof. We will show that $\log(y_{j,n}/Y(j,n))$ is almost surely bounded, which implies that the sequence $\{Y(j, \cdot)\}$ is tight and that there is a δ such that for all n, $Y(j,n) > \delta y_{j,n}$ with probability at least δ. Observe first that the probability of buyer $(n+1)$ attaching to seller j may be bounded below by ignoring the contribution from the α chance of sampling uniformly, and by sampling with replacement on edges not adjacent to seller j but without replacement for seller j. Again, let $Y(n) = Y(j,n)$ for some fixed value of j and let $y_j = E[Y(n)]$. This leads to

$$\Pr(Y(n+1) > Y(n)) \geq \frac{(1-\alpha)\nu Y(n)}{(1+\nu)n} - O\left(\frac{Y(n)^2}{n^2}\right).$$

Now define $Z_n := (n^\beta/j^\beta)(y_j/Y(n))$. Now we may estimate

$$\begin{aligned}
Z_n^{-1} E(Z_{n+1} \mid Z_n) &= \frac{(n+1)^\beta}{n^\beta} E\left(\frac{Y(n)}{Y(n+1)} \mid Y(n)\right) \\
&= \frac{(n+1)^\beta}{n^\beta} E\left(\frac{Y(n+1) - I_{\{Y(n+1) > Y(n)\}}}{Y(n+1)} \mid Y(n)\right) \\
&= \left(1 + \frac{\beta}{n} + O\left(\frac{1}{n^2}\right)\right)\left(1 - \frac{\Pr(Y(n+1) > Y(n) \mid Y(n))}{Y(n+1)}\right) \\
&\leq \left(1 + \frac{\beta}{n} + O\left(\frac{1}{n^2}\right)\right) \\
&\quad \left[1 - \left(\beta\frac{Y(n)}{n} + O\left(\frac{Y(n)^2}{n^2}\right)\right)\left(\frac{1}{Y(n)} - \frac{1}{Y(n)(Y(n)+1)}\right)\right] \\
&= 1 + \frac{\beta}{n} - \frac{\beta}{n} + O(n^{-2}) + \frac{\beta}{nY(n)} + O\left(\frac{Y(n)}{n^2}\right). \\
&= 1 + O\left(\frac{Y(n)}{n^2}\right) + O\left(\frac{1}{nY(n)}\right).
\end{aligned}$$

Let $p_n = \Pr(Y_{n+1} > Y_n)$. By Line 2.1 we know that $p_n = \Theta(Y(n)/n)$. Since each seller attaches to one buyer when it is added to the graph, $Y(n) \geq 1$. This implies that $p_n \geq c/n$ for some constant $c > 0$. Thus we can apply a sharper version of the Borel-Cantelli lemma [31] and conclude that $Y_n \geq c' \log n$, for some constant $c' > 0$. This, in turn, implies that $p_n \geq c \log n/n$. Again applying the stronger version of Borel-Cantelli shows that $Y(n) = \sum_{i=1}^{n} p_i \geq \sum_{i=1}^{n} c' \log n/n = \Omega(\log^2 n)$. By Lemma A.0.1, almost sure boundedness of $\log Z_n$ now follows from almost sure summability of $n^{-1}Y(n)^{-1}$ and $n^{-2}Y(n)$. □

One further extension we need is proved along entirely analogous lines, so we relegate its proof to the appendix (see Section A).

Lemma 2.5.3. *There are constants $c_{\nu,\alpha,p}$ such that for all n, j,*

$$E[Y(n)^p] \leq c_{\nu,\alpha,p} E[Y(n)]^p = O\left(\frac{n}{j}\right)^{p\beta}.$$

2.6 Results: Social Networks

In this section we consider bipartite exchange economies where the underlying graph is generated via the bipartite (α, ν)-model. We will provide two main theorems which describe the relationship between equilibrium wealth and statistical properties of the network. First, we will state and prove a result that shows the distribution of wealth is upper bounded by a power law distribution. Second, we will give a result that bounds the wealth variation (ratio of maximum to minimum wealth) in terms of the parameters α and ν. We will also support these theorems with simulations which exhibit how tight the bounds our theorems provide are. In these simulations, equilibrium computations were done using the algorithm of [27] (or via the application of this algorithm to local subgraphs). We note that it was only the recent development of this algorithm and related ones that made possible the simulations described here (involving hundreds of buyers and sellers in highly cyclical graphs). However, even the speed of this algorithm limits our experiments to networks with $n = 250$ if we wish to run repeated trials to reduce variance. Many of our results suggest that the local approximation method, discussed in Section 2.6.3, may be far more effective. Figure 2.1 and its caption provide an example of a bipartite exchange economy generated by the bipartite (α, ν)-model, along with a discussion of its equilibrium properties.

2.6.1 Wealth Distribution

Theorem 2.6.1. *In the bipartite (α, ν)-model, the proportion of sellers with wealth greater than ω is $O(\omega^{-1/\beta})$. For example, if $\alpha = 0$ (pure preferential attachment) and $\nu = 1$, the proportion falls off as $1/\omega^2$.*

Proof. Fix n and consider the proportion of sellers whose degree exceeds some value, ω. Let j solve $y_{j,n} = \omega$, by Lemma 2.5.3 we get that $j = O(n\omega^{-1/\beta})$. It follows from Lemma 2.5.2 that a positive fraction of sellers arriving by time δj will have degree exceeding ω, proving that the proportion of sellers with degrees exceeding ω is $\Omega(\omega^{-1/\beta})$ as soon as n is big enough so that $n\omega^{-1/\beta} \to \infty$. On the other hand, for $k > j$ we have

$$\begin{aligned}
\Pr(Y(k,n) \geq \omega) &\leq \frac{E[Y(k,n)^p]}{\omega^p} \\
&= O\left(\frac{j}{k}\right)^{\beta p}
\end{aligned}$$

by Lemma 2.5.3. Choosing $p > \beta^{-1}$ and summing over k shows that the expected number of these sellers with degrees exceeding ω is $O(j)$. This shows that the proportion of sellers at time n whose degree is exceeds ω is $\Theta(\omega^{-1/\beta})$. Applying the Frontier Bound (Theorem 2.4.3) completes the proof. □

We do not yet have such a closed-form lower bound on the cumulative price distribution. However, the price distributions seen in large simulation results do indeed show power-law behavior. Interestingly, this occurs despite the fact that degree is a *poor* predictor of *individual* seller price, as we shall see in Section 2.6.3 Figure 2.2 shows empirical *cumulative* price and degree distributions on a loglog scale.

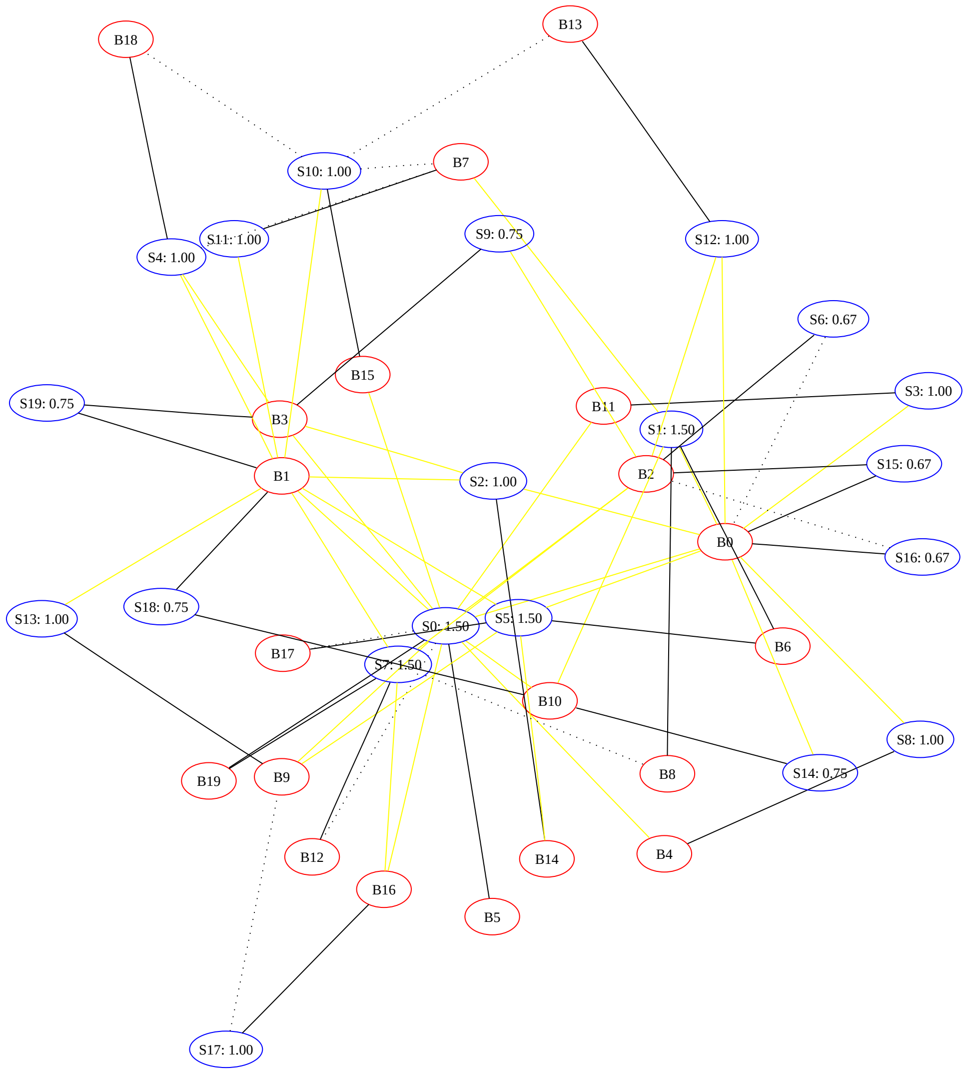

Figure 2.1: Sample bipartite exchange economy generated by the bipartite ($\alpha = 0, \nu = 2$)-model. Buyers and sellers are labeled by 'B' or 'S' respectively, followed by an index indicating the time step at which they were introduced to the network. The solid black edges in the figure show the *exchange subgraph* — those pairs of buyers and sellers who actually exchange currency and goods at equilibrium. The solid yellow edges are edges of the network that are unused at equilibrium because they represent inferior prices for the buyers, while the dotted edges are edges of the network that have competitive prices, but are unused at equilibrium due to the specific consumption plan required for market clearance. Each seller is labeled with the price they charge at equilibrium. The example exhibits non-trivial price variation (from 2.00 down to 0.33 per unit good). Note that while there appears to be a correlation between seller degree and price, it is far from a deterministic relation, a topic we shall examine later.

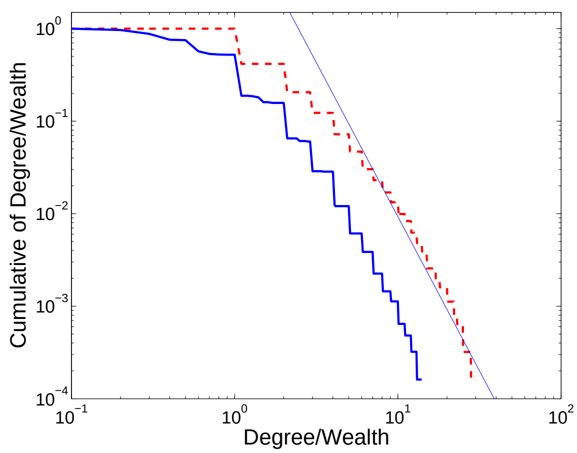

Figure 2.2: Cumulative price and degree distributions on a loglog scale, averaged over 25 networks drawn according to the bipartite ($\alpha = 0.4, \nu = 1$)-model with $n = 250$. The cumulative degree distribution is shown as a dotted line, where the y-axis represents the fraction of the sellers with degree greater than or equal to d, and the degree d is plotted on the x-axis. Similarly, the solid curve plots the fraction of sellers with price greater than some value w, where the price w is shown on the x-axis. The thin sold line has our theoretically predicted slope of $\frac{-1}{\beta} = -3.33$, which shows that degree distribution is quite consistent with our expectations, at least in the tails.

Perhaps the most interesting finding is that the tail of the *price* distribution looks linear, *i.e.* it also exhibits power law behavior. Our theory provided an upper bound, which is precisely the cumulative degree distribution. This plot further confirms the robustness of the power law behavior in the tail, for $\alpha < 1$ and $\nu = 1$.

As discussed in the introduction, Pareto's original observation [91] was that the wealth (which corresponds to seller price in our model) distribution in societies obey a power law, especially in the tails. This observation has been born out in many studies on western economies [29, 30, 84]. Since Pareto's original observation, there have been too many explanations of this phenomena to recount here. However, to our knowledge, all of these explanations are more *dynamic* in nature (*e.g.* a dynamical system of wealth exchange) and do not capture microscopic properties of individual rationality. Here we have power law wealth distribution arising from the combination of certain natural statistical properties of the network, and classical theories of economic equilibrium.

2.6.2 Wealth Variation

Another quantity of interest is what we might call wealth variation — the ratio of the wealth of the richest seller (or buyer) to the poorest seller (or buyer). The following theorem addresses this for the case of sellers.

Theorem 2.6.2. *In the bipartite (α, ν)-model, if $\alpha(\nu^2+1) < 1$, then the ratio of the maximum seller price to the minimum seller price scales with number of buyers n as $\Omega(n^{\frac{2-\alpha(\nu^2+1)}{1+\nu}})$. For the simplest case in which $\alpha = 0$ and $\nu = 1$, this lower bound is just $\Omega(n)$.*

We prove this theorem by again combining the Frontier Bound (Theorem 2.4.3) along with the results on the statistical properties of the network.. This proof is more involved, so we just sketch it here.

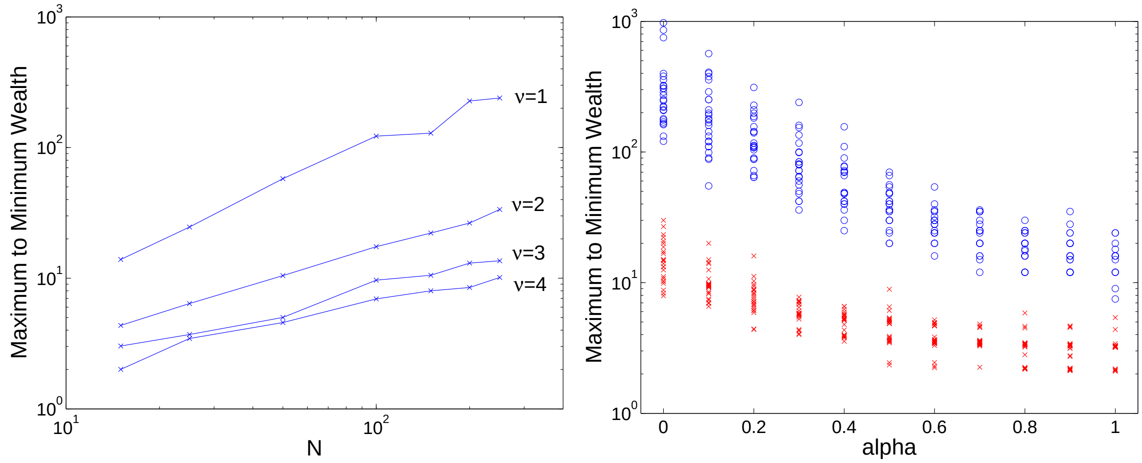

Figure 2.3: The left panel shows the maximum to minimum seller price as function of n (averaged over 25 trials) on a loglog scale. Each line represents a fixed value of ν, between 1 and 4 ($\alpha = 0$). Theorem 2.6.2 predicts the slopes of the lines to be $(1, 0.67, 0.5, 0.4)$. The estimated slopes are somewhat close: $(1.02, 0.71, 0.57, 0.53)$. The right panel is a scatter plot of α vs. the maximum to minimum seller price in a graph (where $n = 250$). Each point represents the price variation in a specific network generated by our model. The circles are for economies generated with $\nu = 1$ and the x's are for economies generated with $\nu = 3$.

Proof. (Sketch) Let us now bound the total wealth of the first ν sellers. By Lemmas 2.5.1 and 2.5.2 the degrees of these sellers at time $n/2$ are all $\Theta(n^{\beta})$, so when a buyer arrives at a time between $n/2$ and n, the probability of one of this buyer's connections links to exactly the first ν sellers is $\Theta(n^{\beta-1})$. Hence, the probability that all of this buyer's connections link to exactly the first ν sellers is $\Theta(n^{\nu\cdot(\beta-1)})$. Summing over the $n/2$ buyers shows that the total number of such buyers is, with high probability, $\Theta(n^{1+\nu\cdot(\beta-1)})$. Deleting from this list those buyers who are later linked by some seller removes a constant fraction of these (details omitted).

Using similar arguments as in the proof of Lemma 2.5.1, one can show the first buyer has degree $\Theta(n^{\frac{1-\alpha}{1+\nu}})$. A similar argument to above shows that the number of sellers *only* connected to this buyer, is $\Omega(n^{\frac{1-\alpha}{1+\nu}})$. Combining this with the previous bound and applying the Frontier Bound (Theorem 2.4.3) leads to the result. $\square$

We now provide a brief examination of how price variation depends on the parameters of the bipartite (α, ν)-model. We first experimentally evaluate the lower bounds provided in Theorem 2.6.2. The left panel of Figure 2.3 shows the maximum to minimum price as function of n. Recall from Theorem 2.6.2, our lower bound on the ratio is $\Omega(n^{\frac{2}{1+\nu}})$ (using $\alpha = 0$). We conjecture that this is tight, and, if so, the slopes of lines (in the loglog plot) should be $\frac{2}{1+\nu}$, which would be $(1, 0.67, 0.5, 0.4)$. The estimated slopes are somewhat close: $(1.02, 0.71, 0.57, 0.53)$. The overall message is that for small values of ν, price variation increases rapidly with the economy size n in preferential attachment.

The rightmost panel of Figure 2.3 is a scatter plot of α vs. the maximum to minimum price in a graph. Here we see that in general, increasing α dramatically decreases price variation (note that the price ratio is plotted on a log scale). This justifies the intuition that as α is increased, more "economic equality" is introduced in the form of less preferential bias in the formation of new edges. Furthermore, the data for $\nu = 1$ shows much larger variation, suggesting that a larger value of ν also has the effect of equalizing buyer opportunities and therefore prices.

2.6.3 Performance of Local Approximation Method

Recall that the Frontier Bound (Theorem 2.4.3) suggests a method by which we can do only *local* computations to approximate the *global* equilibrium price for any seller. More precisely, for some seller j, consider the subgraph which contains all nodes that are within distance k of j. In our bipartite setting, for k odd, this subgraph has a buyer frontier, and for k even, this subgraph has a seller frontier, since we start from a seller. Hence, by the Frontier Bound the equilibrium computation on the odd k (respectively, even k) subgraph will provide an upper (respectively, lower) bound.

This provides an heuristic in which one can examine the equilibrium properties of small regions of the graph, without having to do expensive global equilibrium computations. The effectiveness of this heuristic will of course depend on how fast

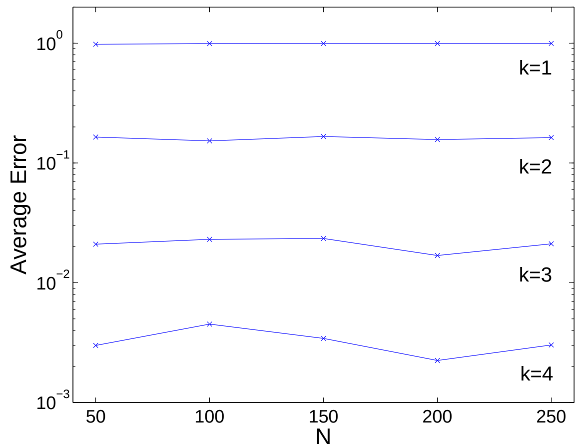

Figure 2.4: In these experiments, graphs were generated by the bipartite ($\alpha = 0, \nu = 1$) model. The value of n is given on the x-axis; the average errors (over 5 trials for each value of k and n) in the local equilibrium computations are given on the y-axis; and there is a separate plot for each of 4 values for k.

the upper and lower bounds tighten. In general, it is possible to create specific graphs in which these bounds are arbitrarily poor until k is large enough to encompass the entire graph. As we shall see, the performance of this heuristic is dramatically better in the bipartite (α, ν)-model.

Figure 2.4 shows how rapidly the local equilibrium computations converge to the true global equilibrium prices as a function of k, and also how this convergence is influenced by n. It appears that for each value of k, the quality of approximation obtained has either mild or no dependence on n. Furthermore, the regular spacing of the four plots on the logarithmic scaling of the y-axis establishes the fact that the error of the local approximations is decaying *exponentially* with increased k — indeed, by examining only neighborhoods of 3 steps from a seller in an economy of hundreds, we are already able to compute approximations to global equilibrium prices with errors in the second decimal place. Since the diameter for $n = 250$ was often about 17, this local graph is considerably smaller than the global. However, for the crudest approximation $k = 1$, which corresponds exactly to using seller degree as a proxy for price, we can see that this performs rather poorly. Computationally, we found that the time required to do all 250 local computations for $k = 3$ was about 60% less than the global computation, and would result in presumably greater savings at much larger values of n.

2.7 An Experimental Illustration on International Trade Data

We conclude with a brief experiment exemplifying some of the ideas discussed so far. The statistics division of the United Nations makes available extensive data sets detailing the amounts of trade between major sovereign nations[6]. We used a data set indicating, for each pair of nations, the total amount of trade in U.S. dollars between that pair in the year 2002.

For our purposes, we would like to extract a discrete network structure from this numerical data. There are many reasonable ways this could be done; here we describe just one. For each of the 70 largest nations (in terms of total trade), we include connections from that nation to each of its top k trading partners, for some integer $k > 1$. We are thus including the more "important" edges for each nation. Note that each nation will have degree at least k, but as we shall see, some nations will have much higher degree, since they frequently occur as a top k partner of other nations. To further cast this extracted network into the bipartite setting we have been considering, we ran many trials in which each nation is randomly assigned a role as either a buyer or seller (which are symmetric roles), and then computed the equilibrium prices of the resulting network economy. We have thus deliberately created an experiment in which *the only economic asymmetries are those determined by the undirected network structure.*

The leftmost panel of Figure 2.5 show results for 1000 trials under the choice $k = 3$. The upper plot shows the average equilibrium price for each nation, where

[6] See http://unstats.un.org/unsd/comtrade

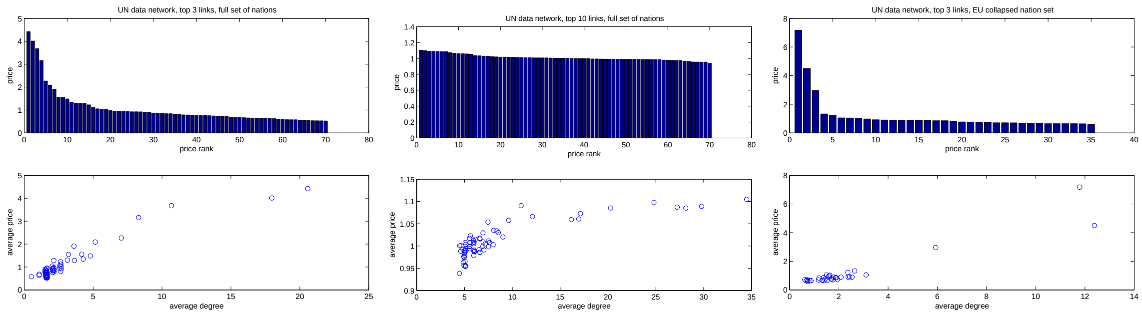

Figure 2.5: Left Column: each nation is connected to its top 3 trading partners. Middle Column: each nation is connected to its top 10 trading partners. Right Column: each nation is connected to its top 3 trading partners with the members of European Union combined into 1 nation.

the nations have been sorted by this average price. We can immediately see that there is dramatic price variation due to the network structure; while many nations suffer equilibrium prices well under \$1, the most topologically favored nations command prices of \$4.42 (U.S.), \$4.01 (Germany), \$3.67 (Italy), \$3.16 (France), \$2.27 (Japan), and \$2.09 (Netherlands). The lower plot of the leftmost panel shows a scatterplot of a nation's degree (x-axis) and its average equilibrium price (y-axis). We see that while there is generally a monotonic relationship, at smaller degree values there can be significant price variation (on the order of \$0.50).

The center panel of Figure 2.5 shows identical plots for the choice $k = 10$. As suggested by the theory and simulations, increasing the overall connectivity of each party radically reduces price variation, with the highest price being just \$1.10 and the lowest just under \$1. Interestingly, the identities of the nations commanding the highest prices (in order, U.S., France, Switzerland, Germany, Italy, Spain, Netherlands) overlaps significantly with the $k = 3$ case, suggesting a certain robustness in the relative economic status predicted by the model. The lower plot shows that the relationship between degree and price divides the population into "have" (degree above 10) and "have not" (degree below 10) components.

The preponderance of European nations among the top prices suggests our final experiment, in which we modified the $k = 3$ network by *merging* the 15 current members of the European Union (E.U.) into a single economic nation. This merged vertex has much higher degree than any of its original constituents and can be viewed as an (extremely) idealized experiment in the economic power that might be wielded by a truly unified Europe.

The rightmost panel of Figure 2.5 provides the results, where we show the relative prices and the degree-price scatterplot for the 35 largest nations. The top prices are now commanded by the E.U. ($7.18), U.S. ($4.50), Japan ($2.96), Turkey ($1.32), and Singapore ($1.22). The scatterplot shows a clear example in which the highest degree (held by the U.S.) does not command the highest price.

Chapter 3

A Network Formation Game for Bipartite Exchange Economies

3.1 Introduction

Recently there has been interest in both the computer science and economics communities in network formation games. Broadly speaking, in these multiplayer games, individuals may choose to share the cost of building a network by purchasing edges incident on themselves. Each player's overall utility consists of two, usually competing, components — on the one hand, the edge costs incurred by the player, and on the other, some measure of "benefit" accrued to the player by their participation or position in the network.

For instance, in one well-studied model [38, 1], individuals wish to minimize their edge purchases plus the sum of their (shortest-path) distances to all other players. Clearly there is a trade-off between these two components. Within such models there have been studies of the structural properties of those networks that are (pure strategy) Nash equilibria of the game, Price of Anarchy bounds, and other analyses (see Section 3.2 for related work).

As in the example above, in much of the prior research the benefit to a player for participating in the network measures some notion of their *centrality* or *connectivity* — shortest-path distances to other players, number of other players in the same component, and so on. In this chapter we introduce and analyze a natural alternative — namely, we view the network formed by the players as defining *trading opportunities*, and measure the network benefit to a player by the wealth they accrue from those trading opportunities.

Our point of departure is a recently introduced networked version of classical exchange economies [55], and more specifically its specialization to bipartite buyer-seller networks discussed in Chapter 2 (which was originally presented in [56]). In the latter, there is an exogenously specified bipartite network between n buyers, who each have an endowment of 1 divisible unit of an abstract commodity called cash, and n sellers, who each have an endowment of 1 divisible unit of an abstract commodity called wheat. Buyers have utility only for wheat and sellers only for cash, thus ensuring mutual interest in trade. The bipartite network is viewed as specifying all and only those pairs of buyers and sellers who may trade. Earlier work [32, 55] established the existence of (market-clearing) equilibria in which prices and wealths may vary across the network due to topological asymmetries, paving the way for the later study presented in Chapter 2 in which the network is generated according to standard stochastic (non-strategic) network generation models. There it was established (for example) that Erdős-Rényi networks exhibit essentially no price or wealth variation, while those generated according to preferential attachment have unbounded wealth variation (growing as a root of the population size).

In this chapter we start with the same bipartite buyer-seller model, but now *endogenize* the creation of the network to arrive at a natural network formation game. More precisely, we assume that any buyer (respectively, seller) is free to purchase an edge to any seller (respectively, buyer) at a cost of α. The selection of which edges to purchase by all parties specifies an undirected bipartite network G, and in this network each party i achieves some exchange equilibrium wealth $\omega(G, i)$.

Our network formation game is then defined by specifying the *overall* utility to i as

$$u_i = -\alpha \times e(G, i) + \omega(G, i)$$

where $e(G, i)$ is the number of edges in G purchased by i. We view the u_i as defining a one-shot, simultaneous move game over the $2n$ players, in which each player's action is a selection of which edges to purchase; see Section 3.3 for a formal definition and discussion of the game.

The network formation game given by the u_i is similar in broad spirit to previous network formation games, but quite different in its details. As with previous models, each player is balancing an outlay of wealth for edge creation (trading opportunities) against some resulting participatory benefit in G; but now the participatory benefit is measured in terms of wealth gained from trade rather than connectivity or shortest paths.

Our main results provide a precise structural characterization of all the networks G that are Nash equilibria of the game defined by the payoffs u_i above. More precisely, we establish exact conditions on the amount of exchange equilibrium wealth variation that can occur for any given values of α and n, and show that this in turn sharply limits the connectivity structure of any Nash equilibrium network G. We then show that these limits are tight by demonstrating specific Nash equilibrium networks G that saturate them, thus yielding a comprehensive catalog of all Nash equilibria. The resulting characterization also places sharp limits on the possible exchange rates or prices that are possible. For example, while with an exogenously specified graph, any rational exchange rate can be achieved, only very specific exchange rates can be achieved in a graph that is a Nash equilibrium of the formation game — an exchange rate of $2/5$, for instance, is impossible.

To our knowledge this is the first network formation game of comparable complexity for which such a complete understanding of its Nash equilibria has been given; for prior models only broad structural restrictions have been established.

3.2 Related Work

Networks formation games have been studied in both the economics and computer science literatures. The structure and characteristics of the networks that arise from such games were first theoretically researched by Aumann and Myerson [5]. For a recent and detailed review of social science and economics models see Jackson [51].

Remaining in the economics literature, but more directly related to our work, Kranton and Minehart [73] also considered bipartite exchange economies and network formation. In their models, buyer valuations are drawn from a known distribution, and the pricing mechanism used is that of a generalized English (ascending-bid) auction. Their main interests were the study of the efficiency of the formed networks and in showing that Nash equilibria networks are efficient; they also characterize Nash equilibrium structure for certain values of the edge cost. In contrast to these works, here we examine exchange equilibrium and provide a complete characterization of all Nash equilibrium networks.

Within computer science, most works have concentrated on network formation routing games, and the main interest has been the quality of the resulting equilibrium, as measured by the price of anarchy and the price of stability. We now survey most of these results.

Anshelevich et al. [3, 2] considered a network formation game in which each player or node is given a set of nodes to which she wishes to connect. Players are allowed to share the cost of an edge and thus may pay for remote edges. In the first work [3], any cost-sharing mechanism was considered and it was proven that there is a pure approximate 3-Nash equilibrium whose cost is that of the social optimum. An efficient algorithm to calculate an efficient 4.65-Nash was also provided. In the second paper [2] only a fair sharing mechanism that uses the Shapely value was considered. The price of anarchy in this setting is trivially $O(n)$, but they discovered that the *price of stability* was $O(\log n)$, and a matching lower bound was provided.

Fabrikant et al. [38], followed by Albres et al. [1], studied a game in which the goal of each player or node is to minimize the sum of distances to the other nodes and his edge costs, where the cost of each edge is α. The main results of these papers prove constant price of anarchy for almost every edge price α. A different variant of this model was studied by Corbo and Parkes [23] where the cost of an edge was shared equally by its endpoints; once again the main interest was in the price of anarchy, not in network structure.

Recently, Moscibroda et al. [81] studied a similar model with applications to peer-to-peer topologies. The goal of each player is to minimize the sum of stretches to other nodes and the edge costs. (The stretch is defined as the distance in the formed graph divided by an initial distance, which is decided according to the input metric). They also study the price of anarchy and the existence of pure Nash equilibrium.

Finally, Johari et al. [53] also considered a routing-based formation game. They considered a directed network, where each node wishes to send a given amount of traffic to other nodes. The cost function for a node/player v is composed from three components: the first is negative and is due to the edges purchased by v; the second is positive and is due to the nodes reachable from v; and the third is negative and is due to the amount of traffic that goes through v. The edges are bought by bilateral negotiation between the endpoints. The main results of [53] provide an equilibrium existence proof and a study of the equilibrium structure conditioned on the payoff function.

3.3 The Network Formation Game

We begin by defining a few concepts related to the bipartite exchange economy model studied in Chapter 2, and then extend this model to our network formation game.

3.3.1 Bipartite Exchange Economies

In order to formally define the network formation game we consider in this chapter, we need the definition of a bipartite exchange economy, which is given in Section 2.3. We also need to define two additional concepts related to the graphical aspects of exchange equilibria. First, observe that in a bipartite exchange economy an exchange equilibrium not only determines the wealth of each player, but the consumption plan also determines on which edges trading takes place. We call the subgraph that consists of edges where trading occurred an *exchange subgraph*.

Definition 3.3.1. *Let $G = (B, S, E)$ be a bipartite exchange economy. Let $\{\omega_i^b\}$, $\{\omega_j^s\}$, and $\{x_{ij}\}$ be an exchange equilibrium, then the exchange subgraph of G is $G' = (B, S, E')$, where $E' = \{(i, j) \mid x_{ij} > 0\}$.*

In contrast to the exchange equilibrium wealth, the exchange subgraph need not be unique. We say that exchange subgraph G' is minimal if the removal of any edge from G' changes the exchange equilibrium wealths. Note that even when G is connected its exchange subgraph may be disconnected. We thus call the connected components of the exchange subgraph *trading components*. We say that a trading component is (n, k) if there are n buyers and k sellers. We will show that this will result in the wealth of each buyer in such component being k/n, and the wealth of each seller being n/k. Thus, wherever there is a wealth variation in G there are at least two trading components that have a different ratio between buyers and sellers in them.

In the bipartite exchange economy model described in Section 2.3, the graph over which the buyers and sellers trade is exogenously defined. That is, the graph is fixed *a priori*, and then the players trade according to it. The main contribution of Chapter 2 is to describe how the topology of the graph affects variation in price of the goods. The main contribution of this chapter is to make the formation of the graph endogenous to the game. That is, players are allowed to buy edges to other players, as opposed to having a topology imposed on them. We now give the formal definition of this new model.

3.3.2 The Network Formation Game

In this section we give a formal definition of the network formation game. This game consists of two sets of players, B and S, where $|B| = |S| = n$. The set B is defined as the buyer set, and the set S is defined as the seller set. As in the bipartite exchange economy we assume that each buyer starts off with an infinitely divisible endowment of 1 unit of an abstract good, which we call cash. Each seller starts off with an infinitely divisible endowment of 1 unit of another abstract good, which we call wheat.

The action of a buyer b_i is denoted $a_i^b \in \{0,1\}^n$ and the action of seller j is denoted $a_j^s \in \{0,1\}^n$. These actions encode which edges, if any, a player buys. An edge (b_i, s_j) is bought by player b_i only if $a_i^b(j) = 1$ and it is bought by s_j only if $a_j^s(i) = 1$. (At equilibrium, an edge (b_i, s_j) will be bought by b_i or s_j or neither, but not both.) A strategy is said to be pure if no player is randomizing over their actions; in this chapter we study only pure strategies. Next, let $a = a_1^b \times \ldots \times a_n^b \times a_1^s \times \ldots \times a_n^s$ be the joint action of all the players. Let the set of edges that b_i buys be denoted $E_i^b(a) = \{(b_i, s_j) \mid a_i^b(j) = 1\}$, and let the set of edges that s_j buys be $E_j^s(a) = \{(b_i, s_j) \mid a_j^s(i) = 1\}$. The joint action of all the players defines a bipartite graph, $G(a) = (B, S, E)$ as follows. The nodes on one side of the graph represent the buyers and on the other side represent the sellers. The set of edges E are the edges that the players bought, or more formally: $E = \bigcup_{i \in [n], t \in \{b,s\}} E_i^t(a)$. Observe that every graph G defines a bipartite exchange economy. We call the price vector and consumption plan that form an equilibrium of the bipartite exchange economy an *exchange equilibrium*. This equilibrium will determine the wealth each player earns; the wealth that buyer b_i earns is denoted $\omega_i^b = \omega_i^b(G)$, and the wealth that seller s_j earns is denoted $\omega_j^s = \omega_i^s(G)$. The wealth each player earns will form the positive component of that players utility function. The negative component will be determined by how many edges each player buys. More formally, we define the utility functions of the players of type $t \in \{b, s\}$ in the network formation game as follows:

$$u_i^t(a) = u_i^t(a_i^t, a_{-i}^t) = \omega_i^t - \alpha|E_i^t|.$$

A joint action $a = a_1^b \times \ldots \times a_n^b \times a_1^s \times \ldots \times a_n^s$ is said to be a *Nash equilibrium* if for every player i we have $u_i^t(a_i^t, a_{-i}^t) \geq u_i^t(\hat{a}_i^t, a_{-i}^t)$ for every action $\hat{a}_i^t$. Since we only consider pure strategies for the players actions, we also only consider pure Nash equilibrium. Thus, each Nash equilibrium strategy a induces a graph, G, which we call an *equilibrium graph*.

Some important comments on this model are in order here. First, the utility functions u_i^t above specify the utilities or payoffs to the players of a standard *one-shot, simultaneous move game*: all players simultaneously choose the set of edges they wish to purchase, which in turn determines G and therefore the utility components ω_i^t. Second, it is important to note that there are *two* distinct equilibrium concepts we shall need to reason about. Our primary interest is in the (pure) Nash equilibrium of the game defined by the u_i^t, which is the network formation game. However, the definition of u_i^t itself involves another equilibrium quantity — namely, the wealth ω_i^t that i receives at *exchange* equilibrium in the *fixed* network G. For clarity we shall always refer to equilibria of the formation game given by the u_i simply as Nash equilibria, and to the latter notion as the *exchange equilibria* for a fixed G. Third, note that the u_i^t treat the initial purchase of edges and the exchange equilibrium wealths as taking place in the same "currency", which differs depending on the type of agent: buyers end with wealth measured in wheat, while sellers end with dollars. We can view this as modeling a central "edge banker" who is willing to extend credit in either currency to the players in order to allow the trade network to be built[1].

[1] If desired, the notion of the edge banker can be made formal and endogenous to the game as a third player type with edges as initial endowments and equal utility for dollars and wheat.

3.4 Summary of Main Results

In this section, we state and discuss our main results; proofs of the theorems are given in Section 3.5. Our first result relates the edge cost α to the minimum exchange equilibrium wealth in any Nash equilibrium graph.

Theorem 3.4.1. *Let G be a Nash equilibrium graph of the network formation game, and let $\omega_{\min}$ be the minimum exchange equilibrium wealth in G of any player. Then $\alpha \geq 1 - \omega_{\min}$, or equivalently, $\omega_{\min} \geq 1 - \alpha$.*

Recalling that the average exchange equilibrium wealth is always 1 (since all endowments are equal), Theorem 3.4.1 states a natural limit on how much exchange equilibrium wealth variation can result from the formation game — the smaller the edge costs α, the more equitable these wealths must be. Great variation in wealths can only arise in the presence of high edge costs. The intuition behind this result is that a player of sufficiently low exchange wealth should be able to find another such player to trade with, with the resulting wealth gain more than covering the edge cost. The proof behind this intuition is somewhat subtle owing to the fragility of exchange equilibria — a small change to the underlying graph may cause large and distant changes to the exchange equilibrium.

Theorem 3.4.2. *Let G be any bipartite graph, and let C be a trading component of G with buyer set $\tilde{B}$ and seller set $\tilde{S}$ such that $|\tilde{B}| = m$ and $|\tilde{S}| = k$, $m > k$. Then there exists a node $s \in \tilde{S}$ and an edge e incident on s such that the removal of e from G decreases the exchange equilibrium wealth of s by at most $1/k$. Furthermore, if G is a Nash equilibrium graph of the network formation game, then $\alpha \leq 1/k$.*

The second claim in Theorem 3.4.2 follows from the first by virtue of the fact that at Nash equilibrium, all of the edges in the trading component C must have been purchased by S' — since $m > k$, the buyers in B' are being "exploited" by the smaller number of sellers in S', and thus have better choices of edge purchases.

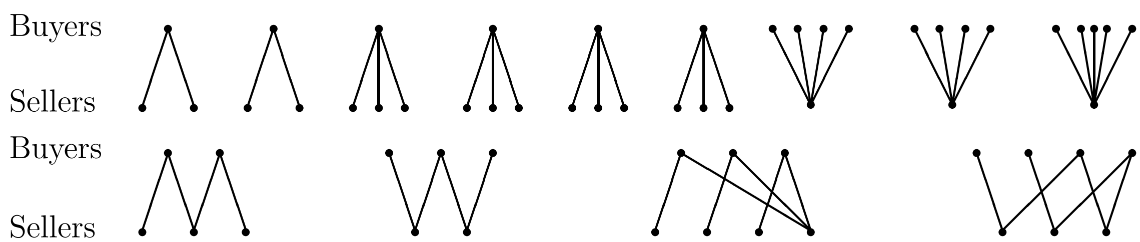

Figure 3.1: Top row: An example of an exploitation graph with $k = 2, \ell = 4$ and $n = 17$. Seller exchange equilibrium wealth values are $1/2, 1/3, 4$ and 5. Bottom row: An example of a balanced graph with $n = 14$. Seller exchange equilibrium wealth values are $2/3, 3/2, 3/4$ and $4/3$.

Together Theorems 3.4.1 and 3.4.2 provide upper and lower bounds on the edge cost α in terms of the minimum exchange wealth and the possible trading component structure. It can be shown that together these bounds strongly constrain the possible Nash equilibrium graphs of the formation game, and that in turn the remaining possibilities can all in fact be realized, leading to a precise characterization of all Nash equilibrium graphs. Before stating our main theorem precisely, we define the following types of graphs.

- *Perfect Matchings.* The class of all perfect matchings between the buyers and sellers. In this class all exchange rates or wealths are equal to 1.

- *Exploitation Graphs.* These are graphs in which *every trading component* has a single party of one type (say, sellers) "exploiting" a (possibly much) larger set of parties of the other type, or vice-versa (a single buyer exploiting many sellers). The collection of such components must meet the constraint that there must be an equal number of buyers and sellers, but also a much stronger constraint on the number of possible different components that can be present simultaneously. More precisely, for any $k, \ell > 1$, let G be a graph consisting of the union of n_1 $(1, k)$-trading components, n_2 $(1, k+1)$-trading components, n_3 $(\ell, 1)$-trading components, and n_4 $(\ell+1, 1)$-trading components, where $n_1 + n_2 + n_3\ell + n_4(\ell+1) = n_1 k + n_2(k+1) + n_3 + n_4$ (equal number of buyers and sellers). Note that in any such graph there may be at most 4 different (say) seller wealth values: $1/k, 1/(k+1), \ell$ and $\ell+1$. Thus for large values of k or ℓ there is great wealth variation. The class of Exploitation Graphs consists of all such graphs G. See Figure 3.1 for an example.

- *Balanced Graphs.* While still permitting some inequality, these graphs are closer to the Perfect Matchings than to the Exploitation Graphs, in that wealth variation is strongly limited. More precisely, for any $k > 2$, let G be a graph consisting of the union of n_1 trading components that are either $(k-1, k)$ or $(k, k+1)$ and n_1 trading components that are either $(k, k-1)$ or $(k+1, k)$, $k > 2$. (Note that since the number of buyers is 1 less than the number of sellers in a $(k-1, k)$-trading component and a $(k, k+1)$-trading component, any mixture of n_1 such components is balanced by any mixture of n_1 $(k, k-1)$ and $(k+1, k)$-trading components.) In such a graph there are again at most 4 different seller wealth values: $k/(k-1), (k+1)/k, (k-1)/k$, and $k/(k+1)$, but unlike in Exploitation Graphs, unbounded wealth variation is not possible, and for large k all wealths are nearly equal. See Figure 3.1 for an example.

Armed with these definitions, we can now state our main theorem, which provides a complete characterization of every Nash equilibria of our network formation game.

Theorem 3.4.3. *Let $NE(n, \alpha)$ be the set of all Nash equilibria graphs of the network formation game for a fixed population size n and edge cost α, and let NE be the union of $NE(n, \alpha)$ over all n and α. Then the set NE equals the union of classes Perfect Matchings, Exploitation Graphs, and Balanced Graphs defined above.*

As has been suggested, the proof that NE is contained in the stated union will follow from Theorems 3.4.1 and 3.4.2 above, while the proof that it contains the union will be shown by explicit construction which is deferred to Appendix B.3. We emphasize that Theorem 3.4.3 places very strong constraints on the Nash equilibrium graphs, and accordingly, on the nature of wealth variation. For instance, the characterization rules out certain exchange rates or wealths — 2/5 is one example of an unattainable value. Wealth variation can essentially occur only in monopolistic form (the exploitation graphs).

Our results rely on one final structural characterization that is of independent interest, and concerns the "compactness" of Nash equilibrium graphs. More precisely, we show that a Nash equilibrium graph G cannot contain redundant edges — that is, the removal of any edge in G will change the exchange subgraph and the exchange rates or wealths. The intuition behind this theorem is that if redundant edges existed, the nodes that purchased them can remove them from the graph without effecting their wealth, and thus it is not a Nash equilibrium. Again there is some subtlety in the proof due to the aforementioned fragility of exchange equilibria. It is interesting to note that in other formation games, such as that in [1], cycles can exist at equilibrium, which can be seen as an analog of redundant edges in our formation game.

Theorem 3.4.4. *Let G be a Nash equilibrium graph of the network formation game. Then G is equal to its minimal exchange subgraph.*

3.5 The Analysis

In this section we provide the proofs of the results described in Section 3.4, although we will defer some of the more technical lemmas to Appendix B. First, we will generalize Theorem 2.4.1 to the case where the number of buyers need not equal the number of sellers. We will then use this result to prove the correctness of a simple algorithm for computing exchange equilibria. Then we will use properties of this algorithm to prove the bounds on α outlined by Theorems 3.4.1 and 3.4.2.

3.5.1 Generalization of Theorem 2.4.1

Theorem 2.4.1 provides a necessary and sufficient condition to have no wealth variation in a graph where the number of sellers equals the number of buyers. Next, we will extend that theorem to graphs where the number of buyers does not equal the number of sellers. This will help us compute the wealth of the sellers in each trading component, since in a given trading component, the number of buyers need not equal the number of sellers. First we will provide a construction that will transform a graph with an unequal number of buyers and sellers to a graph with an equal number of buyers and sellers. Then, we show that the transformed graph has a perfect matching if and only if the original graph has no wealth variation.

Definition 3.5.1. *Let $G = (B, S, E)$ be a bipartite graph such that $|B| = n$ and $|S| = m$. Its τ-*balanced graph *$G' = (B', S', E')$ is constructed as follows. For each $b_i \in B$ make m/τ copies in B', call them $b^i_1, \ldots, b^i_{m/\tau}$, and for each $s_j \in S$ make n/τ copies of it in S', call them $s^j_1, \ldots, s^j_{n/\tau}$. Finally, for each edge $(b_i, s_j) \in E$, add edges to E' to form the complete bipartite graph between $b^i_1, \ldots, b^i_{m/\tau}$ and $s^j_1, \ldots, s^j_{n/\tau}$.*

Lemma 3.5.1. *Let $G = (B, S, E)$ be a bipartite graph such that $|B| = n$ and $|S| = m$. Let $\tau > 0$ be the maximum number such that each element of an exchange equilibrium consumption plan $\{x_{ij}\}$ can be represented as $k\tau$ for an integer k.*[2] *Let $G' = (B', S', E')$ be the τ-balanced graph of G. Then there exists an exchange equilibrium consumption plan for G, where the buyers all earn wealth m/n and the sellers all earn wealth n/m, if and only if G' has a perfect matching.*

We defer the proof of this lemma to the appendix, see Section B.1.

[2]Since the utilities and endowments of the players are rational, the values of the consumption plan are also rational [27]. Thus, such a τ must exist.

Input : $G_1 = (B_1, S_1, E)$ a bipartite exchange economy
Output: the trading components of G_1
$i = 1$;
repeat
Let $U_i = \text{argmax}_{U \subseteq B_i} \frac{|U|}{|N(U)|}$;
$C_i = \{U_i, N(U_i)\}$;
$B_{i+1} = B_i \setminus U_i$, $S_{i+1} = S_i \setminus N(U_i)$;
$E_{i+1} = E_i \setminus \{(u, v) \mid u \in U_i \text{ or } v \in N(U_i)\}$;
$i = i + 1$;
$G_i = (B_i, S_i, E_i)$;
until $B_i = \emptyset$;

Algorithm 1: This algorithm takes as input a bipartite exchange economy, G_1, and outputs the trading components, $C_1, ..., C_r$ of G_1.

3.5.2 The Structure of Nash Equilibria of the Formation Game

The proofs of our main results use an algorithm for determining the trading components of a bipartite exchange economy. For a bipartite graph $G = (B, S, E)$ if W is a set of nodes from one side of the bipartition, then $N(W)$ denotes the set of nodes connected by an edge to some node in W. Algorithm 1 (see Figure) works by iteratively choosing the subset of buyers, $U \subseteq B$ that maximizes $|U|/|N(U)|$, outputs U and $N(U)$, removes them from the graph, and repeats. Intuitively the set of buyers, U, that maximizes this ratio will be getting fairly low wealth since there are many buyers connected to only a few sellers in $N(U)$. Furthermore, buyers not in U that are attached to the sellers in $N(U)$ will likely buy from other sellers since the price in $N(U)$ will be relatively high. There are more general and more efficient algorithms for the equilibrium computation performed by Algorithm 1, such as the algorithm given by Devanur et al. [27]. The simplicity and properties of Algorithm 1, as we shall demonstrate, make it ideal for our structural analysis of the formation game Nash equilibria.

Theorem 3.5.1. *If Algorithm 1 is given any bipartite exchange economy G, then* *it will output all of the trading components of G (which comprises the exchange subgraph of G), along with the wealth of each buyer and seller in G. Furthermore, the connected components output by the algorithm are sorted according to the buyers' wealth in non-decreasing order, i.e.,* $\frac{|U_i|}{|N(U_i)|} \geq \frac{|U_{i+k}|}{|N(U_{i+k})|}$, *for* $k > 0$.

This theorem is proved via the following two lemmas. The first shows that if the players in a bipartite exchange economy trade within the trading components output by Algorithm 1, then there will be no wealth variation within each trading component.

Lemma 3.5.2. *If Algorithm 1 is run on a bipartite exchange economy* $G = (B, S, E)$, *and the players trade within the trading components output,* $C_1 = \{U_1, N(U_1)\}, \ldots, C_r = \{U_r, N(U_r)\}$, *then every* C_i *will have no wealth variation, furthermore, the wealth of each seller in* C_i *will be* $|U_i|/|N(U_i)|$, *and the wealth of each buyer in* C_i *will be* $|N(U_i)|/|U_i|$.

Proof. Let $\tau > 0$ be the maximum number such that each element of an exchange equilibrium consumption plan $\{x_{ij}\}$ can be represented as $k\tau$ for an integer k. Let $H = (U_i, N(U_i), E_i)$, where $E_i = \{(u, v) \mid u \in U_i, v \in N(U_i)$, and $(u, v) \in E\}$. Let $H' = \{U_i', V_i', E_i'\}$ be the τ-balanced graph of H. Assume for the sake of contradiction that there is wealth variation in H. Then, by Lemma 3.5.1 there is no perfect matching in H'. Next, by Theorem 2.4.1 if there is no perfect matching in H', then there is a set $W \subset U_i'$, such that $|N(W)| < |W|$. By the construction of H' we can assume, without loss of generality, that if one of the nodes corresponding to b_i in H' is in W, then all of the nodes corresponding to b_i in H' are in W. More formally, if $b_k^i \in W$ then $\{b_k^i\}_{k=1}^{m/\tau} \subseteq W$. Now let $R(W) = \{b_i \mid b_k^i \in W$ for some $k\}$, then we get the following. (The left most equality comes from the construction of H'.)

$$\begin{aligned} \frac{|R(W)|}{|N(R(W))|} &= \frac{\tau|W|/|N(U_i)|}{\tau|N(W)|/|U_i|} = \frac{|W|}{|N(W)|}\frac{|U_i|}{|N(U_i)|} \\ &> \frac{|U_i|}{|N(U_i)|} \end{aligned}$$

This contradicts the fact that $|U_i|/|N(U_i)|$ has the maximum ratio of buyers to sellers in G_{i-1}. Thus there is no wealth variation in H. Since there is no wealth variation in H at exchange equilibrium, and because of the market clearing condition, the wealth of each seller must be $|U_i|/|N(U_i)|$, and the wealth of each buyer must be $|N(U_i)|/|U_i|$ $\square$

This next lemma establishes the fact that as Algorithm 1 runs, the ratio of the size of the subsets of buyers to the size of their neighbor sets, $|U_i|/|N(U_i)|$, is non-increasing. This is essential to the proof of correctness of our algorithm, because the algorithm assumes that U_i and $N(U_i)$ will form a trading component, and this result shows that neither set will have better trading opportunities.

Lemma 3.5.3. *For any run of Algorithm 1, $|U_i|/|N(U_i)| \geq |U_{i+k}|/|N(U_{i+k})|$ for $k > 0$.*

Proof. Assume for the sake of contradiction that the lemma does not hold, then there exists two consecutive sets such that $|U_i|/|N(U_i)| < |U_{i+1}|/|N(U_{i+1})|$. Consider the set $U_i \cup U_{i+1}$ during a run of Algorithm 1 just before U_i and $N(U_i)$ were removed from the graph.

$$\frac{|U_i \cup U_{i+1}|}{|N(U_i \cup U_{i+1})|} \geq \frac{|U_i| + |U_{i+1}|}{|N(U_i)| + |N(U_{i+1})|} > \frac{|U_i|}{|N(U_i)|},$$

which contradicts the maximality of U_i. $\square$

As consequence of lemmas 3.5.2 and 3.5.3, the proof of Theorem 3.5.1 follows. A key implication of this Theorem is an analog for Lemma 2.4.3.

Corollary 3.5.1. *Let $C = (\tilde{B}, \tilde{S})$. Then C is a trading component if and only if for every subset $B' \subseteq \tilde{B}$, we have $\frac{|B'|}{|N(B')|} \leq \frac{|\tilde{B}|}{|N(\tilde{B})|}$.*

A symmetric claim holds for the sellers.

After showing that the algorithm indeed computes both the exchange equilibrium prices and the exchange subgraph in which they occur, we would like to prove the theorems stated in Section 3.4 using the algorithm's properties. Note that although the proofs rely on Algorithm 1, the statements are independent of the algorithms used to compute the exchange equilibria. The proof of Theorem 3.4.4, which states that the equilibrium graph equals its minimal exchange subgraph, is deferred to Appendix B.4. Next we prove Theorem 3.4.1, which states that α is lower bounded by 1 minus the minimum wealth.

Theorem (3.4.1)**.** *Let G be a Nash equilibrium graph of the network formation game, and let $\omega_{\min}$ be the minimum exchange equilibrium wealth in G of any player. Then $\alpha \geq 1 - \omega_{\min}$, or equivalently, $\omega_{\min} \geq 1 - \alpha$.*

Let u be the buyer who earns the least wealth in G, and let v be the seller who earns the least wealth in G. This proof analyzes how much u would gain if it bought an edge to v. Thus we add the edge (u, v) to G, forming a new graph G'. Running Algorithm 1 on G' allows us to show that u and v would each earn a wealth of 1 in G'. Then, since G is an equilibrium graph and u did not buy an edge to v in G, it must have been the case that $\alpha \geq 1 - \omega_{\min}$.

Proof. Let $C_1 = \{U_1, N(U_1)\}, \ldots, C_r = \{U_r, N(U_r)\}$ denote the connected components output by Algorithm 1 on input G (since G is an equilibrium graph of the network formation game, by Theorem 3.4.4, the union of the connected components equals G itself). Without loss of generality assume that a buyer achieves the minimal wealth. Also, assume that $|U_1| > |N(U_1)|$, otherwise there is no wealth variation and the bound is trivially satisfied, and let $|U_1| = |N(U_1)| + k$, where $k > 0$. By Theorem 3.5.1, $\frac{|U_r|}{|N(U_r)|} = \min_{i \in [r]} \frac{|U_i|}{|N(U_i)|}$, which means that the sellers in C_r get the lowest wealth of all the sellers in G. Now assume $u \in U_1$ connects to $v \in N(U_r)$, and call the resulting graph G'. We now focus our attention on the set $U_1^- = U_1 \setminus \{u\}$. By Theorem 3.5.1, U_1 achieves the maximum ratio of buyers to sellers in G implying $|N(U_1^-)| = |N(U_1)|$, and thus $|U_1^-| = |N(U_1^-)| + k - 1$. More generally, since $\frac{|U_1|}{|N(U_1)|} = \max_{i \in [r]} \frac{|U_i|}{|N(U_i)|}$ in G, for every $W \subset U_1$, $|W| \leq |N(W)| + k - 1$.

Our next step is to run Algorithm 1 on G'. Consider the iteration in which the last part of U_1^- is removed. Let W' be the subset of U_1^- removed in all previous iterations. We have already shown that $|W'| \leq |N(W')| + k - 1$, and subtracting this from $|U_1^-| = |N(U_1^-)| + k - 1$ implies $|U_1^- \setminus W'| \geq |N(U_1^- \setminus W')|$. Thus, by Corollary 3.5.1, the buyers in the the last part of U_1^- earn wealth at most 1. Furthermore, by Theorem 3.5.1, all buyers removed up to this point earn wealth at most 1. Observe that buyer u could not be removed as a part of previous trading component with buyer wealth strictly smaller than 1, as u itself would have added one node to the set, and v would have added one node to the neighbor set, thereby decreasing the ratio of buyers to sellers. Therefore, we can assume that either u has not been removed up to this iteration, or it has been removed with wealth exactly 1. We next show that if it was not removed yet, it will be removed with wealth 1. For any set W that does not contain u and that $v \in N(W)$, we have $|W|/|N(W)| < 1$ (otherwise v would have had a higher wealth in G). Since after the removal of U_1^-, we have $|\{u\}|/|N(\{u\})| = 1$ and this is u's only edge remaining, u and v will be removed together and the wealth of each will be 1. Therefore, the wealth of u would increase by $1 - w_{\min}$ if it bought the edge to v. Since G is an equilibrium graph, this implies $\alpha \geq 1 - w_{\min}$. □

Note that although the proof is referring to an equilibrium graph, we can deduce from the proof the fact that every node with wealth less than 1 can achieve wealth 1 by buying an additional edge. We also note that the following proposition regarding the identities of the players buying the edges follows from the same line of argument.

Theorem 3.5.2. *Let G be a Nash equilibrium graph of the network formation game. Then the exchange equilibrium wealth of each node which buys an edge is at least* 1.

We now go on and prove the upper bound theorem, Theorem 3.4.2, which states that if an (m, k) trading component $(k < m)$ is part of an equilibrium graph, then α is at most $1/k$.

Theorem (3.4.2)**.** *Let G be any bipartite graph, and let C be a trading component of G with buyer set $\tilde{B}$ and seller set $\tilde{S}$ such that $|\tilde{B}| = m$ and $|\tilde{S}| = k$, $m > k$. Then there exists a node $s \in \tilde{S}$ and an edge e incident on s such that the removal of e from G decreases the exchange equilibrium wealth of s by at most $1/k$. Furthermore, if G is a Nash equilibrium graph of the network formation game, then $\alpha \leq 1/k$.*

This proof works by forming a new graph G' which consists of G with a specific type of edge incident on s, removed from it. We run Algorithm 1 on G' and show that s will be removed as part of a set with a ratio of buyers to sellers at least $\frac{m-1}{k}$.

Proof. Let $B' \subset \tilde{B}$ be a strict subset of buyers in C that maximizes the ratio of buyers to neighboring sellers, and let β be this ratio. More formally, let

$$\beta = \underset{X:X\subset\tilde{B} \text{ and } Y=N(X)}{\text{argmax}} \frac{|X|}{|Y|},$$

and let B' and $S' = N(B')$ be sets that maximize the above ratio. We proceed by showing the existence of a seller in S' that can remove one of its edges and decrease its wealth by at most $1/k$. There exists at least one seller $s \in S'$ that has a neighbor $b \notin B'$, because if this were not the case C would not be connected. Now we consider the value of (s, b) to s. Let G' be the graph after the removal of (s, b). Run Algorithm 1 on G', and let $M_1 = (U_1, V_1), ..., M_l = (U_l, V_l)$ be the sets removed by the algorithm before s is removed (i.e. $s \in V_{l+1}$). We will show that after the removal of these sets, the wealth of s is at least $\frac{m-1}{k}$, i.e. $\frac{|U_{l+1}|}{|V_{l+1}|}$ is at least $\frac{m-1}{k}$. In the next few steps we use the following notation: $S_i = S' \cap V_i$, $B_i = B' \cap U_i$, $\bar{S}_l = \cup_{i=1}^{l} S_i$ and $\bar{B}_l = \cup_{i=1}^{l} B_i$. That is, S_i and B_i are the vertices of S' and B' that Algorithm 1 removed during iteration i, and $\bar{S}_l$ and $\bar{B}_l$ are all of the vertices of S' and B' removed through iteration l. Let $x \in \bar{B}_l$, thus $x \in B_i$ for some $i \in [l]$, furthermore $x \in B'$. Let y be any neighbor of x. Since $N(B') = S'$, $y \in S'$. Now $y \in N(B_j)$ for some $1 \leq j \leq i$, thus $y \in S_j$, and so $y \in \bar{W}_l$. Thus with respect to G, $N(\bar{B}_l) \subseteq \bar{S}_l$. By the definition of S' and B', for every $l > 1$, $\frac{|\bar{B}_l|}{|\bar{S}_l|} \leq \beta$.

Partition B' into two sets, $B' \setminus \bar{B}_l$ and $\bar{B}_l$. We have just shown $\beta = \frac{|B'|}{|S'|} \geq \frac{|\bar{B}_l|}{|N(\bar{B}_l)|}$, then we must have $\frac{|B' \setminus \bar{B}_l|}{|N(B' \setminus \bar{B}_l)|} \geq \frac{|B'|}{|S'|} = \beta$. Therefore, when v is removed, it is part of a set with a ratio of buyers to sellers least β which implies (as Algorithm 1 chooses the set that maximizes the ratio) that the set in which it is actually removed with has a ratio at least β as well (note that it is not necessarily $S' \setminus \bar{S}_l$), and thus the wealth of v is at least β. By Lemma B.2.1 the decrease in the wealth of v is at most $m/k - \beta \leq 1/k$, which concludes the first part of Theorem. Furthermore, if the trading component is part of an equilibrium graph of the network formation game, then by Theorem 3.5.2, v buys all of her incident edges and thus $\alpha \leq 1/k$. $\square$

We are finally ready to prove Theorem 3.4.3 which states that the set of Nash equilibrium of the formation game equals the union of the three following graphs families: Perfect Matching, Exploitation, or Balanced. First we show that the set of Nash Equilibrium of the formation game are contained in one of the three families. Then we show that each of these graph families contain Nash equilibria of the formation game.

Theorem (3.4.3)**.** *Let $NE(n, \alpha)$ be the set of all Nash equilibria graphs of the network formation game for a fixed population size n and edge cost α, and let NE be the union of $NE(n, \alpha)$ over all n and α. Then the set NE equals the union of classes Perfect Matchings, Exploitation Graphs, and Balanced Graphs defined above.*

Proof. Let C be an (m, k) trading component of the graph. We next show that only a few values of (m, k) can occur. By Theorem 3.4.1 we have that $\alpha \geq 1 - \frac{k}{m}$. By Theorem 3.4.2 we have that $\alpha \leq 1/k$. Combining these two inequalities we have that

$$1/k \geq 1 - k/m \Rightarrow m \geq k(m - k)$$

which holds only for $k = 1$ and $k = m - 1$. Therefore, the only possible trading components are $(1, k)$ and $(k, k+1)$. First we show that a $(1, k)$ trading component cannot coexist with a $(k, k+1)$ trading component (unless $k = 2$). The first type of trading component implies that $\alpha \geq 1 - 1/k$, and the second implies that $\alpha \leq 1/\ell$, and this can only hold for $\ell = k = 2$, however we defined the $(1,2)$-trading component as an Exploitation$(1,2)$ rather than a balanced graph.

Now consider the case where $(1,k)$ trading component exists along with $(1,k+\ell)$ for $\ell \geq 2$, and let u be the sole buyer in the $(1,k)$ trading component. Now u can buy an edge to a seller v in the $(1,k+\ell)$ component; it is easy to see that now v will trade with u and will earn $1/(k+1)$ wealth, rather than trading inside the $k+\ell$ component and earning wealth $1/(k+\ell)$. Therefore, $(1,k)$ and $(1,k+\ell)$ components cannot both be part of an equilibrium graph of the network formation game. It sill remains to show that $(k,k+1)$ cannot coexist with $(\ell,\ell+1)$. Let us see what are the restrictions imposed by such component. By Theorem 3.4.2 and Theorem 3.4.1, we have that

$$\frac{1}{k+1} \leq \alpha \leq \frac{1}{k}$$

This immediately implies that only consecutive trading components, that is, $(k-1,k)$ and $(k,k+1)$, can coexist. Thus we have shown that every Nash equilibrium of the formation game is either a Perfect Matching, Exploitation, or Balanced graph.

Now we show that each of these 3 families contain Nash equilibrium of the network formation game. Consider a perfect matching graph. Any node that does not buy an edge cannot change its wealth by buying additional edges. If the node did so, the graph would still contain a perfect matching, and by Theorem 2.4.1 the exchange equilibrium would still occur along this perfect matching. Consider a node that buys an edge and deviates to a strategy in which it removes its edge and buys ℓ edges. In this case, this node would have a wealth of $\frac{\ell}{\ell+1}$ and thus would have no incentive to deviate in this manner. If a node, which buys an edge, buys an additional ℓ edges, as before the graph would still contain a perfect matching and the exchange equilibrium would not be changed. By Lemma B.3.2, Exploitation(k,ℓ) graphs are Nash equilibrium for $\alpha > 1-2/(\max(k+1,\ell+1))^2$, and by Lemma B.3.5, Balanced$(k,k+1)$ graphs are Nash equilibrium of the game for $\alpha = 1/(k+1)$. $\square$

Chapter 4

Networks Preserving Evolutionary Equilibria and the Power of Randomization

4.1 Introduction

In this chapter, we introduce and examine a natural extension of classical evolutionary game theory (EGT) to a setting in which pairwise interactions are restricted to the edges of an undirected graph or network. This extension generalizes the classical setting, in which all pairs of organisms in an infinite population are equally likely to interact. The classical setting can be viewed as the special case in which the underlying network is a clique.

There are many obvious reasons why one would like to examine more general graphs, the primary one being in that many scenarios considered in evolutionary game theory, all interactions are in fact not possible. For example, geographical restrictions may limit interactions to physically proximate pairs of organisms. Also, the social structure of a group of organisms may dictate which pairs of organisms interact. More generally, as evolutionary game theory has become a plausible model not only for biological interaction, but also economic and other kinds of interaction in which certain dynamics are more imitative than optimizing (see [10, 94, 49] and chapter 4 of [106]), the network constraints may come from similarly more general sources. Evolutionary game theory on networks has been considered before, but not in the generality we will do so here (see Section 4.4).

We generalize the definition of an evolutionary stable strategy (ESS) to networks, and show a pair of complementary results that exhibit the power of randomization in our setting: subject to degree or edge density conditions, the classical ESS of any 2-player, symmetric game are preserved when the graph is chosen randomly and the mutation set is chosen adversarially, or when the graph is chosen adversarially and the mutation set is chosen randomly. We also prove converses to these main results that show *only* the classical ESS are preserved in these graphical settings. These results show that insofar as classifying stable strategies, the random pairwise matching scheme of the classical model is equivalent to either randomizing the graph or randomizing the mutations. We will see that the reason we can classify the stable strategies in these types of graphs, is that these topologies preclude enclaves of mutants that have lots of internal, mutant-mutant interactions and few external, incumbent-mutants interactions. This is yet another example of a result that shows how the topology of the network can effect the structure of equilibria. We also examine natural strengthenings of our generalized ESS definition, and show that similarly strong results are not possible for them.

The work described here is part of recent efforts examining the relationship between graph topology or structure and properties of equilibrium outcomes. Previous works in this line include studies of the relationship of topology to properties of correlated equilibria in graphical games [54], and studies of price variation in graph-theoretic market exchange models (see Chapter 2). More generally, this work contributes to the line of graph-theoretic models for game theory investigated in both computer science [60] and economics [51].

4.2 Classical EGT

The fundamental concept of evolutionary game theory is the evolutionarily stable strategy (ESS). Intuitively, an ESS is a strategy such that if all the members of a population adopt it, then no mutant strategy could invade the population [96]. To make this more precise, we describe the basic model of evolutionary game theory, in which the notion of an ESS resides.

The classical model of evolutionary game theory considers an infinite population of organisms, where each organism is assumed to be equally likely to interact with each other organism. Interaction is modeled as playing a fixed, 2-player, symmetric game defined by a fitness function F (we emphasize that the same game F is played in all interactions). Let A denote the set of actions available to both players, and let $\Delta(A)$ denote the set of probability distributions or mixed strategies over A, then $F : \Delta(A) \times \Delta(A) \rightarrow \Re$. If two organisms interact, one playing strategy s and the other playing strategy t, the s-player earns a fitness of $F(s|t)$ while the t-player earns a fitness of $F(t|s)$.

In this infinite population of organisms, suppose that there is a $1-\epsilon$ fraction who play strategy s, and call these organisms *incumbents*; and suppose that there is an ϵ fraction who play t, and call these organisms *mutants*. Assume that two organisms are chosen uniformly at random to play each other. The strategy s is an ESS if the expected fitness of an organism playing s is higher than that of an organism playing t, for all $t \neq s$ and all sufficiently small ϵ. Since an incumbent will meet another incumbent with probability $1-\epsilon$ and it will meet a mutant with probability ϵ, we can calculate the expected fitness of an incumbent, which is simply $(1-\epsilon)F(s|s)+\epsilon F(s|t)$. Similarly, the expected fitness of a mutant is $(1-\epsilon)F(t|s) + \epsilon F(t|t)$. Thus we come to the formal definition of an ESS [106].

Definition 4.2.1. *A strategy s is an* evolutionarily stable strategy (ESS) *for the 2-player, symmetric game given by fitness function F, if for every strategy $t \neq s$, there*

	H	D
H	$(V-C)/2$	V
D	0	$V/2$

Figure 4.1: The game of Hawks and Doves

exists an ϵ_t *such that for all* $0 < \epsilon < \epsilon_t$, $(1-\epsilon)F(s|s)+\epsilon F(s|t) > (1-\epsilon)F(t|s)+\epsilon F(t|t)$.

A consequence of this definition is that for s to be an ESS, it must be the case that $F(s|s) \geq F(t|s)$, for all strategies t. This inequality means that s must be a best response to itself, and thus any ESS strategy s must also be a Nash equilibrium. In general the notion of ESS is more restrictive than Nash equilibrium, and not all 2-player, symmetric games have an ESS.

Next, we give an example of a 2-player, symmetric game called Hawks and Doves, and then exhibit its ESS. The game of Hawks and Doves models two organisms fighting over a resource. Obtaining the resource results in a fitness gain of V, while fighting for the resource and losing results in a fitness decrease of C. If a Hawk plays a Dove, the Hawk will fight for the resource and the Dove will give up. This results in a Hawk earning an increase of fitness of V, and the Dove's fitness staying the same. If two Doves play each other, they split the resource earning them both a fitness increase of $V/2$. If two Hawks play, eventually one will win and one will lose, and it assumed that each organism has a 1/2 chance of being the winner. Figure 4.1 shows the payoff matrix for this game.

The strategy profile (D, D) is not a Nash Equilibrium because one player could unilaterally deviate and play H and increase its payoff from $V/2$ to V. Since (D, D) is not a Nash Equilibrium, D cannot be an ESS. Now, if $V > C$ then H is an ESS. To see this observe that $F(H|H) = (V-C)/2$. Let t be any mixed strategy with probability $p < 1$ of playing H and $1-p$ of playing D, then $F(t|H) = p\frac{V-C}{2}+(1-p)0 < (V-C)/2$. Since $F(H|H) > F(t|H)$ for all $t \neq H$, H is an ESS. We leave it as an exercise for the reader to show that if $V \leq C$, the mixed strategy of playing H with probability V/C and D with probability $1 - V/C$ is an ESS. Observe that as $V \to C$, the probability of playing H approaches 1. This coincides with the pure strategy ESS of playing H when $V > C$.

In this chapter our interest is to examine what kinds of network structure *preserve* the ESS strategies for those games that do have a standard ESS. First we must of course generalize the definition of ESS to a network setting.

4.3 EGT on Graphs

In our setting, we will no longer assume that two organisms are chosen uniformly at random to interact. Instead, we assume that organisms interact only with those in their local neighborhood, as defined by an undirected graph or network. As in the classical setting (which can be viewed as the special case of the complete network or clique), we shall assume an infinite population, by which we mean we examine limiting behavior in a family of graphs of increasing size.

Before giving formal definitions, some comments are in order on what to expect in moving from the classical to the graph-theoretic setting. In the classical (complete graph) setting, there exist many symmetries that may be broken in moving to the network setting, at both the group and individual level. Indeed, such asymmetries are the primary interest in examining a graph-theoretic generalization.

For example, at the group level, in the standard ESS definition, one need not discuss any particular set of mutants of population fraction ϵ. Since all organisms are equally likely to interact, the survival or fate of any specific mutant set is identical to that of any other. In the network setting, this may not be true: some mutant sets may be better able to survive than others due to the specific topologies of their interactions in the network. For instance, foreshadowing some of our analysis, if s is an ESS but $F(t|t)$ is much larger than $F(s|s)$ and $F(s|t)$, a mutant set with a great deal of "internal" interaction (that is, edges between mutants) may be able to survive, whereas one without this may suffer. At the level of individuals, in the classical setting, all incumbents have the same fitness and all mutants have the same fitness. Thus the assertion that one mutant dies implies that all mutants die, again by symmetry. In the network setting, individual fates may differ within a group all playing a common strategy. These observations imply that in examining ESS on networks we face definitional choices that were obscured in the classical model.

If G is a graph representing the allowed pairwise interactions between organisms (vertices), and u is a vertex of G playing strategy s_u, then the fitness of u is given by

$$F(u) = \frac{\sum_{v \in \Gamma(u)} F(s_u|s_v)}{|\Gamma(u)|}.$$

Here s_v is the strategy being played by the neighbor v, and $\Gamma(u) = \{v \in V : (u, v) \in E\}$. One can view the fitness of u as the average fitness u would obtain if it played each of its neighbors, or the expected fitness u would obtain if it were assigned to play one of its neighbors chosen uniformly at random.

Classical evolutionary game theory examines an infinite, symmetric population. Graphs or networks are inherently finite objects, and we are specifically interested in their asymmetries, as discussed above. Thus all of our definitions shall revolve around an infinite family $G = \{G_n\}_{n=0}^{\infty}$ of finite graphs G_n over n vertices, but we shall examine asymptotic (large n) properties of such families.

We first give a definition for a family of mutant vertex sets in such an infinite graph family to *contract*.

Definition 4.3.1. *Let $G = \{G_n\}_{n=0}^{\infty}$ be an infinite family of graphs where G_n has n vertices. Let $M = \{M_n\}_{n=0}^{\infty}$ be any family of subsets of vertices of the G_n such that $|M_n| \geq \epsilon n$ for some constant $\epsilon > 0$. Suppose all the vertices of M_n play a common (mutant) strategy t, and suppose the remaining vertices in G_n play a common (incumbent) strategy s. We say that M_n* contracts *if for sufficiently large n, for all but $o(n)$ of the $j \in M_n$, j has an incumbent neighbor i such that $F(j) < F(i)$.*

A reasonable alternative would be to ask that the condition above hold for *all* mutants rather than all but $o(n)$. Note also that we only require that a mutant have *one* incumbent neighbor of higher fitness in order to die; one might consider requiring more. In Sections 4.6.1 and 4.6.2 we consider these stronger conditions and demonstrate that our results can no longer hold.

To properly define an ESS for an infinite family of finite graphs in a way that recovers the classical definition asymptotically in the case of the family of complete graphs, we first must give a definition that restricts attention to families of mutant vertices that are smaller than some invasion threshold $\epsilon' n$, yet remain some constant fraction of the population. This prevents "invasions" that survive merely by constituting a vanishing fraction of the population.

Definition 4.3.2. *Let $\epsilon' > 0$, and let $G = \{G_n\}_{n=0}^{\infty}$ be an infinite family of graphs where G_n has n vertices. Let $M = \{M_n\}_{n=0}^{\infty}$ be any family of (mutant) vertices in G_n. We say that M is* ϵ'-linear *if there exists an ϵ, $\epsilon' > \epsilon > 0$, such that for all sufficiently large n, $\epsilon' n > |M_n| > \epsilon n$.*

We can now give our definition for a strategy to be evolutionarily stable when employed by organisms interacting with their neighborhood in a graph.

Definition 4.3.3. *Let $G = \{G_n\}_{n=0}^{\infty}$ be an infinite family of graphs where G_n has n vertices. Let F be any 2-player, symmetric game for which s is a strategy. We say that s is an ESS with respect to F and G if for all mutant strategies $t \neq s$, there exists an $\epsilon_t > 0$ such that for any ϵ_t-linear family of mutant vertices $M = \{M_n\}_{n=0}^{\infty}$ all playing t, for n sufficiently large, M_n contracts.*

Thus, to *violate* the ESS property for G, one must witness a family of mutations M in which each M_n is an arbitrarily small but nonzero constant fraction of the population of G_n, but does not contract (i.e. every mutant set has a subset of linear size that survives all of its incumbent interactions). In the proof of Theorem 4.5.2 we show that the definition given coincides with the classical one in the case where G is the family of complete graphs, in the limit of large n. We note that even in the classical model, small sets of mutants were allowed to have greater fitness than the incumbents, as long as the size of the set was $o(n)$ [100].

In the definition above there are three parameters: the game F, the graph family G and the mutation family M. Our main results will hold for *any* 2-player, symmetric game F. We will also study two rather general settings for G and M: that in which G is a family of random graphs and M is arbitrary, and that in which G is nearly arbitrary and M is randomly chosen. In both cases, we will see that, subject to conditions on degree or edge density (essentially forcing connectivity of G but not much more), for *any* 2-player, symmetric game, the ESS of the classical settings, and only those strategies, are always preserved. Thus a common theme of these results is the power of randomization: as long as either the network itself is chosen randomly, or the mutation set is chosen randomly, classical ESS are preserved.

4.4 Related Work

There has been previous work that analyzes which strategies are resilient to mutant invasions with respect to various types of graphs. What sets our work apart is that the model we consider encompasses a significantly more general class of games and graph topologies. Moreover, all of the work described below analyzes the limiting behavior of specific dynamics of a population, whereas we seek to describe the relationship between topology and equilibrium behavior by comparing the classical notion of an ESS to the graphical notion. We will briefly survey this literature and point out the differences in the previous models and ours.

In [35], [11], and [12], the authors consider specific families of graphs, such as cycles and lattices, where players play specific games, such as 2×2-games or $k \times k$-coordination games. In these papers the authors specify a simple, local dynamic for players to improve their payoffs by changing strategies, and analyze what type of strategies will grow to dominate the population. The model we propose is more general than both of these, as it encompasses a larger class of graphs as well as a richer set of games.

Also related to our work is that of [75], where the authors propose two models. The first assumes organisms interact according to a weighted, undirected graph. However, the fitness of each organism is simply assigned and does not depend on the actions of each organism's neighborhood. The second model has organisms arranged around a directed cycle, where neighbors play a 2×2-game. With probability proportional to its fitness, an organism is chosen to reproduce by placing a replica of itself in its neighbors position, thereby "killing" the neighbor. We consider more general games than the first model and more general graphs than the second.

Finally, the following papers are most closely related to our work: [34, 80, 21, 86]. The authors consider 2-action, coordination games played by players in a general undirected graph. In these three works, the authors specify a dynamic for a strategy to reproduce, and analyze properties of the graph that allow a strategy to overrun the population. Here again, one can see that our model is more general than these, as it allows for organisms to play any 2-player, symmetric game.

4.5 Networks Preserving ESS

We now proceed to state and prove two complementary results in the network ESS model defined in Section 4.3. First, we consider a setting in which the graphs are generated via the $G_{n,p}$ model of Erdős and Rényi [13]. In this model, every pair of vertices is joined by an edge independently and with probability p (where p may depend on n). The mutant set, however, will be constructed adversarially (subject to the linear size constraint given by Definition 4.3.3). For these settings, we show that for any 2-player, symmetric game, s is a classical ESS of that game, if and only if s is an ESS for $\{G_{n,p}\}_{n=0}^{\infty}$, where $p = \Omega(1/n^c)$ and $0 \leq c < 1$, and any mutant family $\{M_n\}_{n=0}^{\infty}$, where each M_n has linear size. We note that under these settings, if we let $c = 1 - \gamma$ for small $\gamma > 0$, the expected number of edges in G_n is $n^{1+\gamma}$ or larger — that is, just superlinear in the number of vertices and potentially far smaller than $O(n^2)$. It is easy to convince oneself that once the graphs have only a linear number of edges, we are flirting with disconnectedness, and there may simply be large mutant sets that can survive in isolation due to the lack of any incumbent interactions in certain games. Thus we examine the minimum plausible edge density.

The second result is a kind of dual to the first, considering a setting where the graphs are chosen arbitrarily (subject to conditions) but the mutant sets are chosen randomly. It states that for any 2-player, symmetric game, s is a classical ESS for that game, if and only if s is an ESS for any $\{G_n = (V_n, E_n)\}_{n=0}^{\infty}$ in which for all $v \in V_n, \deg(v) = \Omega(n^\gamma)$ (for any constant $\gamma > 0$), and a family of mutant sets $\{M_n\}_{n=0}^{\infty}$, that is chosen randomly (that is, in which each organism is labeled a mutant with constant probability $\epsilon > 0$). Thus, in this setting we again find that classical ESS are preserved subject to edge density restrictions. Since the degree assumption is somewhat strong, we also prove another result which only assumes that $|E_n| \geq n^{1+\gamma}$, and shows that there must exist at least 1 mutant with an incumbent neighbor of higher fitness (as opposed to showing that all but $o(n)$ mutants have an incumbent neighbor of higher fitness). As will be discussed, this rules out "stationary" mutant invasions.

4.5.1 Random Graphs, Adversarial Mutations

Now we state and prove a theorem which shows that if s is a classical ESS, then s will be an ESS for random graphs where a linear sized set of mutants is chosen by an adversary.

Theorem 4.5.1. *Let F be any 2-player, symmetric game, and suppose s is a classical ESS of F. Let the infinite graph family $G = \{G_n\}_{n=0}^{\infty}$ be drawn according to $G_{n,p}$, where $p = \Omega(1/n^c)$ and $0 \leq c < 1$. Then with probability 1, s is an ESS with respect to F and G.*

The main idea of the proof is to divide mutants into two categories, those with "normal" fitness and those with "abnormal" fitness. Here normal fitness will mean within a $(1 \pm \tau)$ factor of the fitness defined by the classical definition of an ESS, and abnormal fitness will mean fitness outside of that range. First, we show all but $o(n)$ of the population (incumbent or mutant) have an incumbent neighbor of normal fitness. This will imply that all but $o(n)$ of the mutants of normal fitness have an incumbent neighbor of *higher* fitness. The vehicle for proving this is Theorem 2.15 of [13], which gives an upper bound on the number of vertices not connected to a sufficiently large set. This theorem assumes that the size of this large set is known with equality which necessitates the union bound argument below. Secondly, we show that there can be at most $o(n)$ mutants with abnormal fitness. Since there are so few of them, even if none of them have an incumbent neighbor of higher fitness, s will still be an ESS with respect to F and G.

Proof. Let $t \neq s$ be the mutant strategy. Since s is a classical ESS, there exists an ϵ_t such that $(1-\epsilon)F(s|s) + \epsilon F(s|t) > (1-\epsilon)F(t|s) + \epsilon F(t|t)$, for all $0 < \epsilon < \epsilon_t$. Let M be any mutant family that is ϵ_t-linear. Thus for any fixed value of n that is sufficiently large, there exists an ϵ such that $|M_n| = \epsilon n$ and $\epsilon_t > \epsilon > 0$. Also, let $I_n = V_n \setminus M_n$ and let $I' \subseteq I_n$ be the set of incumbents that have fitness in the range $(1 \pm \tau)[(1-\epsilon)F(s|s) + \epsilon F(s|t)]$ for some constant τ, $0 < \tau < 1/6$. Lemma 4.5.1 below shows $(1-\epsilon)n \geq |I'| \geq (1-\epsilon)n - O\left(\frac{\log n}{p}\right)$. Finally, let

$$T_{I'} = \{x \in V \setminus I' : \Gamma(x) \cap I' \neq \emptyset\}.$$

(For the sake of clarity we suppress the subscript n on the sets I' and T.) The union bound gives us

$$\Pr(|T_{I'}| \geq \delta n) \leq \sum_{i=(1-\epsilon)n - O\left(\frac{\log n}{p}\right)}^{(1-\epsilon)n} \Pr(|T_{I'}| \geq \delta n \text{ and } |I'| = i) \tag{4.1}$$

Letting $\delta = n^{-\gamma}$ for some $\gamma > 0$ gives $\delta n = o(n)$. We will apply Theorem 2.15 of [13] to the summand on the right hand side of Equation 4.1, in order to do so, we will next check that we meet all of the conditions of this theorem.

First we show $\delta pn \geq 3 \log n$. Observe that $\delta pn = \Omega(n^{-\gamma} n^{-c} n) = \Omega(n^{1-c-\gamma})$. Letting $\gamma = (1-c)/2$, combined with the fact that $0 \leq c < 1$, ensures $1 - c - \gamma > 0$ and $\gamma > 0$. Next, in theorem 2.15, $C = pi$ and we need to check that $C \geq 3 \log(e/\delta)$. Since $\delta = n^{-(1-c)/2}$, $3 \log(e/\delta) = O(\log n)$. From the lower bound on i given by equation 4.1, $C \geq (1-\epsilon)np - O(\log n) = \Omega(n^{1-c})$. Since $0 \leq c < 1$ we have satisfied this requirement. Finally, we need to check that $\lim_{n\to\infty} C\delta n = \infty$.

$$\begin{aligned} C\delta n &\geq [(1-\epsilon)np - O(\log n)]\, n^{-(1-c)/2} n \\ &= n^{1/2+c/2} \left[(1-\epsilon)n^{1-c} - O(\log n)\right] \qquad (4.2) \end{aligned}$$

Since $0 \leq c < 1$, the last line above is $\omega(n^{1/2})$. This is the last condition of Theorem 2.15 [13]. When we apply this theorem to equation 4.1, we get

$$\begin{aligned} \Pr(|T_{I'}| \geq \delta n) &\leq \sum_{i=(1-\epsilon)n - O\left(\frac{\log n}{p}\right)}^{(1-\epsilon)n} \exp\left(-\frac{1}{6} C\delta n\right) \qquad (4.3) \\ &\leq O\left(\frac{\log n}{p}\right) \exp\left(-\frac{1}{6} n^{\frac{1+c}{2}} \left[(1-\epsilon)n^{1-c} - O(\log n)\right]\right) \qquad (4.4) \\ &= o(n) \exp(-\omega(n^{1/2})) \qquad (4.5) \\ &= o(1) \end{aligned}$$

Line 4.4 comes from the previously established lower bound on $C\delta n$ given by Line 4.2. Line 4.5 combines $p = \Omega(1/n^c)$ and $0 \leq c < 1$ to show that $O\left(\frac{\log n}{p}\right) = o(n)$, and it also uses the previously established fact that the quantity on the right of Line 4.4 is $\exp(-\omega(n^{1/2}))$. Thus we have shown, with probability tending to 1 as $n \to \infty$, at most $o(n)$ individuals are not attached to an incumbent which has fitness in the range $(1 \pm \tau)[(1-\epsilon)F(s|s) + \epsilon F(s|t)]$. This implies that the number of mutants of approximately normal fitness, not attached to an incumbent of approximately normal fitness, is also $o(n)$.

Now those mutants of approximately normal fitness that *are* attached to an incumbent of approximately normal fitness have fitness in the range $(1 \pm \tau)[(1 - \epsilon)F(t|s) + \epsilon F(t|t)]$. The incumbents that they are attached to have fitness in the range $(1\pm\tau)[(1-\epsilon)F(s|s)+\epsilon F(s|t)]$. Since s is an ESS of F, we know $(1-\epsilon)F(s|s)+\epsilon F(s|t) > (1 - \epsilon)F(t|s) + \epsilon F(t|t)$, thus if we choose τ small enough, we can ensure that all but $o(n)$ mutants of normal fitness have a neighboring incumbent of higher fitness.

Finally by Lemma 4.5.1, we know there are at most $O\left(\frac{\log n}{p}\right) = o(n)$ mutants of abnormal fitness. So even if all of them are more fit than their respective incumbent neighbors, we have shown all but $o(n)$ of the mutants have an incumbent neighbor of higher fitness. □

We now state and prove the lemma used in the proof above.

Lemma 4.5.1. *For almost every graph $G_{n,p}$ with $(1-\epsilon)n$ incumbents, all but $O\left(\frac{\log n}{p}\right)$ incumbents have fitness in the range $(1\pm\tau)[(1-\epsilon)F(s|s)+\epsilon F(s|t)]$, where $p = \Omega(1/n^c)$ and ϵ, τ and c are constants satisfying $0 < \epsilon < 1$, $0 < \tau < 1/6$, $0 \leq c < 1$. Similarly, under the same assumptions, all but $O\left(\frac{\log n}{p}\right)$ mutants have fitness in the range $(1 \pm \tau)[(1 - \epsilon)F(t|s) + \epsilon F(t|t)]$.*

Proof. We define the *mutant degree* of vertex v to be the number of mutant neighbors of v which we denote by $\deg_M(v)$. Similarly, we define the *incumbent degree* of a vertex v to be the number of incumbent neighbors of v which we denote by $\deg_I(i)$. (Note that $\deg(v) = \deg_I(v)+\deg_M(v)$.) Theorem 2.14 of [13] states that the number of incumbents with mutant degree outside the range $(1 \pm \delta)p|M|$ is at most $\frac{12 \log n}{\delta^2 p}$. By the same theorem, the number of incumbents with incumbent degree outside the range $(1 \pm \delta)p|I|$ is at most $\frac{12 \log n}{\delta^2 p}$. Let S be the set of incumbents with either incumbent degree outside the range $(1 \pm \delta)p|I|$ or mutant degree outside the range $(1 \pm \delta)p|M|$ or both. We have shown that $|S| \leq \frac{24 \log n}{\delta^2 p} = O\left(\frac{\log n}{p}\right)$. Let $i \in I \setminus S$. Next, we analyze the average expected fitness of i.

$$F(i) = \frac{\deg_I(i)}{\deg(i)} F(s|s) + \frac{\deg_M(i)}{\deg(i)} F(s|t)$$

where,

$$\begin{aligned}
\frac{\deg_I(i)}{\deg(i)} &\in \frac{(1 \pm \delta)p|I|}{(1 \pm \gamma)pn} = \frac{(1 \pm \delta)(1 - \epsilon)}{1 \pm \gamma} \\
\frac{\deg_M(i)}{\deg(i)} &\in \frac{(1 \pm \delta)p|M|}{(1 \pm \gamma)pn} = \frac{(1 \pm \delta)\epsilon}{1 \pm \gamma}
\end{aligned}$$

Above we made use Corollary 3.14 of [13], which states that if $pn/\log n \to \infty$ then the maximum degree of a vertex in almost every graph is $\{1 + o(1)\}pn$. We also used Lemma C.0.1, which states that if $p = \Omega(1/n^c)$ for some constant $0 \leq c < 1$, then the minimum degree of a vertex in almost every G_p is at least $\{1 - \gamma\}pn$, for all constants $\gamma > 0$. Thus,

$$F(i) = \frac{(1 \pm \delta)}{(1 \pm \gamma)} [(1 - \epsilon)F(s|s) + \epsilon F(s|t)]$$

So we can choose δ and γ small enough such that $F(i) = (1 \pm \tau)[(1 - \epsilon)F(s|s) + \epsilon F(s|t)]$. The proof for the mutant case is analogous. $\square$

Theorem 4.5.3 states that if s is a classical ESS and $G = \{G_{n,p}\}$, where $p = \Omega(1/n^c)$ and $0 \leq c < 1$, then with probability 1 as $n \to \infty$, s is an ESS with respect to G. Here we show that if s is an ESS with respect to G, then s is a classical ESS. In order to prove this theorem, we do not need the full generality of s being an ESS for G when $p = \Omega(1/n^c)$ where $0 \leq c < 1$. All we need is s to be an ESS for G when $p = 1$. In this case there are no more probabilistic events in the theorem statement. Also, since $p = 1$ each graph in G is a clique, thus all of the incumbents will have identical fitness and all of the mutants will have identical fitness. So if one incumbent has a higher fitness than one mutant, then all incumbents have higher fitness than all mutants. This gives rise to the following theorem.

Theorem 4.5.2. *Let F be any 2-player, symmetric game, and suppose s is a strategy for F and $t \neq s$ is a mutant strategy. Let $G = \{K_n\}_{n=0}^{\infty}$. If, as $n \to \infty$, for any*

ϵ_t-linear family of mutants $M = \{M_n\}_{n=0}^{\infty}$, there exists an incumbent i and a mutant j such that $F(i) > F(j)$, then s is a classical ESS of F.

The proof of this theorem analyzes the limiting behavior of the mutant population as the size of the cliques in G tends to infinity. It also shows how the definition of ESS given in Section 4.5 recovers the classical definition of ESS.

Proof. Since each graph in G is a clique, every incumbent will have the same number of incumbent and mutant neighbors, and every mutant will have the same number of incumbent and mutant neighbors. Thus, all incumbents will have identical fitness and all mutants will have identical fitness. Next, one can construct an ϵ_t-linear mutant family M, where the fraction of mutants converges to ϵ for any ϵ, where $\epsilon_t > \epsilon > 0$. So for n large enough, the number of mutants in K_n will be arbitrarily close to ϵn. Thus, *any* mutant subset of size ϵn will result in all incumbents having fitness $(1 - \frac{\epsilon n}{n-1})F(s|s) + \frac{\epsilon n}{n-1}F(s|t)$, and all mutants having fitness $(1 - \frac{\epsilon n - 1}{n-1})F(t|s) + \frac{\epsilon n - 1}{n-1}F(t|t)$. Furthermore, by assumption the incumbent fitness must be higher than the mutant fitness. This implies,

$$\lim_{n\to\infty}\left(\left(1 - \frac{\epsilon n}{n-1}\right)F(s|s) + \frac{\epsilon n}{n-1}F(s|t) > \left(1 - \frac{\epsilon n - 1}{n-1}\right)F(t|s) + \frac{\epsilon n - 1}{n-1}F(t|t)\right)$$

This implies, $(1-\epsilon)F(s|s) + \epsilon F(s|t) > (1-\epsilon)F(t|s) + \epsilon F(t|t)$, for all ϵ, where $\epsilon_t > \epsilon > 0$. □

4.5.2 Adversarial Graphs, Random Mutations

We now move on to our second main result. Here we show that if the graph family, rather than being chosen randomly, is arbitrary subject to a minimum degree requirement, and the mutation sets are randomly chosen, classical ESS are again preserved. A modified notion of ESS allows us to considerably weaken the degree requirement to a minimum edge density requirement.

Theorem 4.5.3. *Let $G = \{G_n = (V_n, E_n)\}_{n=0}^{\infty}$ be an infinite family of graphs in which for all $v \in V_n$, $\deg(v) = \Omega(n^{\gamma})$ (for any constant $\gamma > 0$). Let F be any 2-player, symmetric game, and suppose s is a classical ESS of F. Let t be any mutant strategy, and let the mutant family $M = \{M_n\}_{n=0}^{\infty}$ be chosen randomly by labeling each vertex a mutant with constant probability ϵ, where $\epsilon_t > \epsilon > 0$. Then with probability 1, s is an ESS with respect to F, G and M.*

Proof. Let $t \neq s$ be the mutant strategy and let X be the event that every incumbent has fitness within the range $(1 \pm \tau)[(1-\epsilon)F(s|s) + \epsilon F(s|t)]$, for some constant $\tau > 0$ to be specified later. Similarly, let Y be the event that every mutant has fitness within the range $(1 \pm \tau)[(1-\epsilon)F(t|s) + \epsilon F(t|t)]$. Since $\Pr(X \cap Y) = 1 - \Pr(\neg X \cup \neg Y)$, we proceed by showing $\Pr(\neg X \cup \neg Y) = o(1)$.

$\neg X$ is the event that there exists an incumbent with fitness outside the range $(1 \pm \tau)[(1-\epsilon)F(s|s) + \epsilon F(s|t)]$. If $\deg_M(v)$ denotes the number of mutant neighbors of v, similarly, $\deg_I(v)$ denotes the number of incumbent neighbors of v, then an incumbent i has fitness $\frac{\deg_I(i)}{\deg(i)} F(s|s) + \frac{\deg_M(i)}{\deg(i)} F(s|t)$. Since $F(s|s)$ and $F(s|t)$ are fixed quantities, the only variation in an incumbents fitness can come from variation in the terms $\frac{\deg_I(i)}{\deg(i)}$ and $\frac{\deg_M(i)}{\deg(i)}$. One can use the Chernoff bound followed by the union bound to show that for any incumbent i,

$$\Pr(F(i) \notin (1 \pm \tau)[(1-\epsilon)F(s|s) + \epsilon F(s|t)]) < 4 \exp\left(-\frac{\epsilon \deg(i) \tau^2}{3}\right).$$

Next one can use the union bound again to bound the probability of the event $\neg X$,

$$\Pr(\neg X) \leq 4n \exp\left(-\frac{d_i \tau^2}{3}\right)$$

where $d_i = \min_{i \in V \setminus M} \deg(i)$, $0 < \epsilon \leq 1/2$. An analogous argument can be made to show $\Pr(\neg Y) < 4n \exp(-\frac{\epsilon d_j \tau^2}{3})$, where $d_j = \min_{j \in M} \deg(j)$ and $0 < \epsilon \leq 1/2$. Thus, by the union bound,

$$\Pr(\neg X \cup \neg Y) < 8n \exp\left(-\frac{\epsilon d \tau^2}{3}\right)$$

where $d = \min_{v \in V} \deg(v)$, $0 < \epsilon \leq 1/2$. Since $\deg(v) = \Omega(n^\gamma)$, for all $v \in V$, and ϵ, τ and γ are all constants greater than 0,

$$\lim_{n \to \infty} \frac{8n}{\exp(\epsilon d \tau^2/3)} = 0,$$

so $\Pr(\neg X \cup \neg Y) = o(1)$. Thus, we can choose τ small enough such that $(1+\tau)[(1-\epsilon)F(t|s)+\epsilon F(t|t)] < (1-\tau)[(1-\epsilon)F(s|s)+\epsilon F(s|t)]$, and then choose n large enough such that with probability $1 - o(1)$, every incumbent will have fitness in the range $(1 \pm \tau)[(1-\epsilon)F(s|s) + F(s|t)]$, and every mutant will have fitness in the range $(1 \pm \tau)[(1-\epsilon)F(t|s) + \epsilon F(t|t)]$. So with high probability, every incumbent will have a higher fitness than every mutant. □

The assumption on the degree of each vertex of Theorem 4.5.3 is rather strong. The following theorem relaxes this requirement and only necessitates that every graph have $n^{1+\gamma}$ edges, for some constant $\gamma > 0$, in which case it shows there will alway be at least 1 mutant with an incumbent neighbor of higher fitness. A strategy that is an ESS in this weakened sense will essentially rule out stable, static sets of mutant invasions, but not more complex invasions. An example of more complex invasions are mutant sets that survive, but only by perpetually "migrating" through the graph under some natural evolutionary dynamics, akin to "gliders" in the well-known Game of Life [9].

Theorem 4.5.4. *Let F be any game, let s be a classical ESS of F, and let $t \neq s$ be a mutant strategy. For any graph family $G = \{G_n = (V_n, E_n)\}_{n=0}^{\infty}$ in which $|E_n| \geq n^{1+\gamma}$ (for any constant $\gamma > 0$), and any mutant family $M = \{M_n\}_{n=0}^{\infty}$ which is determined by labeling each vertex a mutant with probability ϵ, where $\epsilon_t > \epsilon > 0$, the probability that there exists a mutant with an incumbent neighbor of higher fitness approaches 1 as $n \to \infty$.*

The main idea behind the proof is to show that with high probability, over only the choice of mutants, there will be an incumbent-mutant edge in which both vertices have high degree. If their degree is high enough, we can show that close to an ϵ fraction of their neighbors are mutants, and thus their fitnesses are very close to what we expect them to be in the classical case. Since s is an ESS, the fitness of the incumbent will be higher than the mutant.

Proof. We call an edge $(i, j) \in E_n$ a $g(n)$-*barbell* if $\deg(i) \geq g(n)$ and $\deg(j) \geq g(n)$. Suppose G_n has at most $h(n)$ edges that are $g(n)$-barbells. This means there are at least $|E_n| - h(n)$ edges in which at least one vertex has degree at most $g(n)$. We call these vertices *light* vertices. Let $\ell(n)$ be the number of light vertices in G_n. Observe that $|E_n| - h(n) \leq \ell(n)g(n)$. This is because each light vertex is incident on at most $g(n)$ edges. This gives us that

$$|E_n| \leq h(n) + \ell(n)g(n) \leq h(n) + ng(n).$$

So if we choose $h(n)$ and $g(n)$ such that $h(n)+ng(n) = o(n^{1+\gamma})$, then $|E_n| = o(n^{1+\gamma})$. This contradicts the assumption that $|E_n| = \Omega(n^{1+\gamma})$. Thus, subject to the above constraint on $h(n)$ and $g(n)$, G_n must contain at least $h(n)$ edges that are $g(n)$-barbells.

Now let H_n denote the subgraph induced by the barbell edges of G_n. Note that regardless of the structure of G_n, there is no reason that H_n should be connected. Thus, let m be the number of connected components of H_n, and let $c_1, c_2, \ldots, c_m$ be the number of vertices in each of these connected components. Note that since H_n is an edge-induced subgraph we have $c_k \geq 2$ for all components k. Let us choose the mutant set by first flipping the vertices in H_n only. We now show that the probability, with respect to the random mutant set, that *none* of the components of H_n have an incumbent-mutant edge is exponentially small in n.

$$\begin{aligned}
\Pr[\text{All components are uniformly labeled}] &= \Pi_{k=1}^{m}(\epsilon^{c_k} + (1-\epsilon)^{c_k}) && (4.6)\\
&= (1-\epsilon)^{\sum_{k=1}^{m} c_k}\Pi_{k=1}^{m}(1+\alpha^{c_k}) && (4.7)\\
&\leq (1-\epsilon)^{\sum_{k=1}^{m} c_k}(1+\alpha^2)^m && (4.8)
\end{aligned}$$

$$
\begin{aligned}
&\leq (1-\epsilon)^{\sum_{k=1}^{m} c_k}(1+(\beta\epsilon)^2)^m && (4.9)\\
&\leq (1-\epsilon)^{\sum_{k=1}^{m} c_k}(1-\epsilon)^{-m\epsilon\beta^2} && (4.10)\\
&\leq (1-\epsilon)^{(1-\frac{\epsilon\beta^2}{2})\sum_{k=1}^{m} c_k} && (4.11)
\end{aligned}
$$

If we let $\alpha = \frac{\epsilon}{1-\epsilon}$, line 4.7 comes from simple algebra. We use the facts that $\alpha < 1$ and $c_k \geq 2$ to arrive at the upper bound in line 4.8. Since $\alpha = \frac{\epsilon}{1-\epsilon}$, for any constant $\beta > 0$, there exists an ϵ sufficiently small, such that $\alpha < \beta\epsilon$, this gives us the bound in line 4.9. Line 4.10 comes from getting like bases for the exponents in line 4.9, and by using Taylor series to show $\log(1+x) = x + O(x^2)$ when $|x| < 1$. Since each component contains at least 1 edge, $\sum_{k=1}^{m} c_k \geq 2m$, using this fact gave us the last line.

We can choose ϵ small enough to ensure that the exponent in line 4.11 is positive. Furthermore, since $\sum_{k=1}^{m} \binom{c_k}{2} \geq h(n)$, we can use Lemma C.0.2 to show that $\sum_{k=1}^{m} c_k \geq \sqrt{h(n)}$. Thus, we can let $h(n) = n^{\gamma/4}$ and the probability that all components are uniformly labeled will go to 0, as n tends to infinity.

Now assuming that there exists a non-uniformly labeled component, by construction that component contains an edge (i, j) where i is an incumbent and j is a mutant, that is a $g(n)$-barbell. We also assume that the $h(n)$ vertices already labeled have been done so arbitrarily, but that the remaining $g(n) - h(n)$ vertices neighboring i and j are labeled mutants independently with probability ϵ. Then via a standard Chernoff bound argument, one can show that with high probability, the fraction of mutants neighboring i and the fraction of mutants neighboring j is in the range $(1 \pm \tau)\frac{(g(n)-h(n))\epsilon}{g(n)}$. Similarly, one can show that the fraction of incumbents neighboring i and the fraction of mutants neighboring j is in the range $1 - (1 \pm \tau)\frac{(g(n)-h(n))\epsilon}{g(n)}$.

Since s is an ESS, there exists a $\zeta > 0$ such that $(1-\epsilon)F(s|s) + \epsilon F(s|t) = (1-\epsilon)F(t|s) + \epsilon F(t|t) + \zeta$. If we choose $g(n) = n^{\gamma}$, and $h(n) = o(g(n))$, we can choose n large enough and τ small enough to force $F(i) > F(j)$, as desired. $\square$

Theorem 4.5.3 states that if s is a classical ESS for a 2-player, symmetric game F, where G is chosen adversarially subject to the constraint that the degree of each vertex is $\Omega(n^{\gamma})$ (for any constant $\gamma > 0$), and mutants are chosen with probability ϵ, then s is an ESS with respect to F, G, and M. Here we show that if s is an ESS with respect to F, G, and M then s is a classical ESS.

All we will need to prove this is that s is an ESS with respect to $G = \{K_n\}_{n=0}^{\infty}$, that is when each vertex has degree $n-1$. As in Theorem 4.5.2, since the graphs are cliques, if one incumbent has higher fitness than one mutant, then all incumbents have higher fitness than all mutants. Thus, the theorem below is also a converse to Theorem 4.5.4. (Recall that Theorem 4.5.4 uses a weaker notion of contraction that requires only one incumbent to have higher fitness than one mutant.)

Theorem 4.5.5. *Let F be any 2-player symmetric game, and suppose s is an incumbent strategy for F and $t \neq s$ is a mutant strategy. Let $G = \{K_n\}_{n=0}^{\infty}$. If with probability 1 as $n \to \infty$, s is an ESS for G and a mutant family $M = \{M_n\}_{n=0}^{\infty}$, which is determined by labeling each vertex a mutant with probability ϵ, where $\epsilon_t > \epsilon > 0$, then s is a classical ESS of F.*

This proof also analyzes the limiting behavior of the mutant population as the size of the cliques in G tends to infinity. One has to show show that as $n \to \infty$, the incumbent fitness converges to $(1-\epsilon)F(s|s) + \epsilon F(s|t)$, and the mutant fitness converges to $(1-\epsilon)F(t|s) + \epsilon F(t|t)$. Observe that the exact fraction mutants of V_n is now a random variable. Since the mutants are chosen randomly we will use an argument similar to the proof that a sequence of random variables that converges in probability, also converge in distribution. In this case the sequence of random variables will be actual fraction of mutants in each K_n. So to prove this convergence we use an argument similar to one that is used to prove that sequence of random variables that converges in probability also converges in distribution (details omitted). Having proved this, we will show that since s is an ESS for $\{K_n\}_{n=0}^{\infty}$, the incumbent fitness must be higher than the mutant fitness. This in turn establishes that s must be a classical ESS, and we thus obtain a converse to Theorem 4.5.3.

Proof. Fix any value of ϵ, where $\epsilon_n > \epsilon > 0$, and construct each M_n by labeling a vertex a mutant with probability ϵ. By the same argument as in the proof of Theorem 4.5.2, if the actual number of mutants in K_n is denoted by $\epsilon_n n$, *any* mutant subset of size $\epsilon_n n$ will result in all incumbents having fitness $(1 - \frac{\epsilon_n n}{n-1})F(s|s) + \frac{\epsilon_n n}{n-1}F(s|t)$, and in all mutants having fitness $(1 - \frac{\epsilon_n n-1}{n-1})F(t|s) + \frac{\epsilon_n n-1}{n-1}F(t|t)$. This implies

$$\lim_{n\to\infty} \Pr(s \text{ is an ESS for } G_n \text{ w.r.t. } \epsilon_n n \text{ mutants}) = 1 \Rightarrow$$

$$\lim_{n\to\infty} \Pr\Big((1 - \frac{\epsilon_n n}{n-1})F(s|s) + \frac{\epsilon_n n}{n-1}F(s|t) > (1 - \frac{\epsilon_n n - 1}{n-1})F(t|s) + \frac{\epsilon_n n - 1}{n-1}F(t|t) \Big) = 1 \Leftrightarrow$$

$$\lim_{n\to\infty} \Pr\left(\epsilon_n > \frac{F(t|s) - F(s|s)}{F(s|t) - F(s|s) - F(t|t) + F(t|s)} + \frac{F(s|s) - F(t|t)}{n}\right) = 1$$

By two simple applications of the Chernoff bound and an application of the union bound (see Lemma C.0.3), one can show the sequence of random variables $\{\epsilon_n\}_{n=0}^{\infty}$ converges to ϵ in probability. Next, if we let $X_n = -\epsilon_n, X = -\epsilon, b = -F(s|s) + F(t|t)$, and $a = -\frac{F(t|s)-F(s|s)}{F(s|t)-F(s|s)-F(t|t)+F(t|s)}$, by Theorem 4.5.6 below, we get that $\lim_{n\to\infty} \Pr(X_n < a + b/n) = \Pr(X < a)$. Combining this with equation 4.12, $\Pr(\epsilon > -a) = 1$. ∎

The proof of the following theorem is very similar to the proof that a sequence of random variables that converges in probability, also converge in distribution. A good explanation of this can be found in [48], which is the basis for the argument below.

Theorem 4.5.6. *If $\{X_n\}_{n=0}^{\infty}$ is a sequence of random variables that converge in probability to the random variable X, and a and b are constants, then* $\lim_{n\to\infty} \Pr(X_n < a + b/n) = \Pr(X < a)$.

Proof. By Lemma 4.5.2 (see below) we have the following two inequalities,

$$\begin{aligned}\Pr(X < a + b/n - \tau) &\leq \Pr(X_n < a + b/n) + \Pr(|X - X_n| > \tau),\\ \Pr(X_n < a + b/n) &\leq \Pr(X < a + b/n + \tau) + \Pr(|X - X_n| > \tau).\end{aligned}$$

Combining these gives,

$$\begin{aligned}\Pr(X < a + b/n - \tau) - \Pr(|X - X_n| > \tau) &\leq \Pr(X_n < a + b/n)\\ &\leq \Pr(X < a + b/n + \tau) +\\ &\quad \Pr(|X - X_n| > \tau).\end{aligned}$$

There exists an n_0 such that for all $n > n_0$, $|b/n| < \tau$, so the following statement holds for all $n > n_0$.

$$\begin{aligned}\Pr(X < a - 2\tau) - \Pr(|X - X_n| > \tau) &\leq \Pr(X_n < a + b/n)\\ &\leq \Pr(X < a + 2\tau) + \Pr(|X - X_n| > \tau).\end{aligned}$$

Take the $\lim_{n\to\infty}$ of both sides of both inequalities, and since X_n converges in probability to X,

$$\Pr(X < a - 2\tau) \leq \lim_{n\to\infty} \Pr(X_n < a + b/n) \tag{4.12}$$

$$\leq \Pr(X < a + 2\tau). \tag{4.13}$$

Recall that X is a continuous random variable representing the fraction of mutants in an infinite sized graph. So if we let $F_X(a) = \Pr(X < a)$, we see that $F_X(a)$ is a cumulative distribution function of a continuous random variable, and is therefore continuous from the right. So

$$\lim_{\tau\downarrow 0} F_X(a - \tau) = \lim_{\tau\downarrow 0} F_X(a + \tau) = F_X(a).$$

Thus if we take the $\lim_{\tau\downarrow 0}$ of inequalities 4.12 and 4.13 we get

$$\Pr(X < a) = \lim_{n\to\infty} \Pr(X_n < a + b/n).$$

□

The following lemma is quite useful, as it expresses the cumulative distribution of one random variable Y, in terms of the cumulative distribution of another random variable X and the difference between X and Y.

Lemma 4.5.2. *If X and Y are random variables, $c \in \Re$ and $\tau > 0$, then*

$$\Pr(Y < c) \leq \Pr(X < c + \tau) + \Pr(|Y - X| > \tau).$$

Proof.

$$\begin{aligned} \Pr(Y < c) &= \Pr(Y < c, X < c + \tau) + \Pr(Y < c, X \geq c + \tau) \\ &\leq \Pr(Y < c \mid X < c + \tau) \Pr(X < c + \tau) + \Pr(|Y - X| > \tau) \\ &\leq \Pr(X < c + \tau) + \Pr(|Y - X| > \tau) \qquad \square \end{aligned}$$

4.6 Limitations of Stronger Models

In this section we show that if one tried to strengthen the model described in Section 4.3 in two natural ways, one would not be able to prove results as strong as Theorems 4.5.1 and 4.5.3, which hold for every 2-player, symmetric game.

4.6.1 Stronger Contraction for the Mutant Set

In Section 4.3 we alluded to the fact that we made certain design decisions in arriving at Definitions 4.3.1, 4.3.2 and 4.3.3. One such decision was to require that all but $o(n)$ mutants have incumbent neighbors of higher fitness. Instead, we could have required that *all* mutants have an incumbent neighbor of higher fitness. The two theorems in this subsection show that if one were to strengthen our notion of contraction for the mutant set, given by Definition 4.3.1, in this way, it would be impossible to prove theorems analogous to Theorems 4.5.1 and 4.5.4.

Recall that Definition 4.3.1 gave the notion of contraction for a linear sized subset of mutants. In what follows, we will say an *edge* (i, j) contracts if i is an incumbent, j is a mutant, and $F(i) > F(j)$. Also, recall that Theorem 4.5.1 stated that if s is a classical ESS, then it is an ESS for random graphs with adversarial mutations. Next, we prove that if we instead required *every* incumbent-mutant edge to contract, this need not be the case.

Theorem 4.6.1. *Let F be a 2-player, symmetric game that has a classical ESS s for which there exists a mutant strategy $t \neq s$ with $F(t|t) > F(s|s)$ and $F(t|t) > F(s|t)$. Let $G = \{G_n\}_{n=0}^{\infty}$ be an infinite family of random graphs drawn according to $G_{n,p}$, where $p = \Omega(1/n^c)$ for any constant $0 \leq c < 1$. Then with probability approaching 1 as $n \to \infty$, there exists a mutant family $M = \{M_n\}_{n=0}^{\infty}$, where $\epsilon_t n > |M_n| > \epsilon n$ and $\epsilon_t, \epsilon > 0$, in which there is an edge that does not contract.*

Proof. (Sketch) With probability approaching 1 as $n \to \infty$, there exists a vertex j where $\deg(j)$ is arbitrarily close to ϵn. So label j mutant, label one of its neighbors incumbent, denoted i, and label the rest of j's neighborhood mutant. Also, label all of i's neighbors incumbent, with the exception of j and j's neighbors (which were already labeled mutant). In this setting, one can show that $F(j)$ will be arbitrarily close to $F(t|t)$ and $F(i)$ will be a convex combination of $F(s|s)$ and $F(s|t)$, which are both strictly less than $F(t|t)$. $\square$

Theorem 4.5.4 stated that if s is a classical ESS, then for graphs where $|E_n| \geq n^{1+\gamma}$, for some $\gamma > 0$, and where each organism is labeled a mutant with probability ϵ, one edge must contract. Below we show that, for certain graphs and certain games, there will always exist one edge that will not contract.

Theorem 4.6.2. *Let F be a 2-player, symmetric game that has a classical ESS s, such that there exists a mutant strategy $t \neq s$ where $F(t|s) > F(s|t)$. There exists an infinite family of graphs $\{G_n = (V_n, E_n)\}_{n=0}^{\infty}$, where $|E_n| = \Theta(n^2)$, such that for a mutant family $M = \{M_n\}_{n=0}^{\infty}$, which is determined by labeling each vertex a mutant* *with probability $\epsilon > 0$, the probability there exists an edge in E_n that does not contract approaches 1 as $n \to \infty$.*

Proof. (Sketch) Construct G_n as follows. Pick $n/4$ vertices $u_1, u_2, \ldots, u_{n/4}$ and add edges such that they from a clique. Then, for each u_i, $i \in [n/4]$ add edges (u_i, v_i), (v_i, w_i) and (w_i, x_i). With probability 1 as $n \to \infty$, there exists an i such that u_i and w_i are mutants and v_i and x_i are incumbents. Observe that $F(v_i) = F(x_i) = F(s|t)$ and $F(w_i) = F(t|s)$. □

4.6.2 Stronger Contraction for Individuals

The model of Section 4.3 requires that for an edge (i, j) to contract, the fitness of i must be greater than the fitness of j. One way to strengthen this notion of contraction would be to require that the *maximum* fitness incumbent in the *neighborhood* of j be more fit than the *maximum* fitness mutant in the *neighborhood* of j. This models the idea that each organism is trying to take over each place in its neighborhood, but only the most fit organism in the neighborhood of a vertex gets the privilege of taking it. If we assume that we adopt this notion of contraction for individual mutants, and require that all incumbent-mutant edges contract, we will next show that Theorems 4.6.1 and 4.6.2 still hold, and thus it is still impossible to get results such as Theorems 4.5.1 and 4.5.4 which hold for every 2-player, symmetric game.

In the proof of Theorem 4.6.1 we proved that $F(i)$ is strictly less than $F(j)$. Observe that maximum fitness mutant in the neighborhood of j must have fitness at least $F(j)$. Also observe that there is only 1 incumbent in the neighborhood of j, namely i. So under this stronger notion of contraction, the edge (i, j) will not contract.

Similarly, in the proof of Theorem 4.6.2, observe that the only mutant in the neighborhood of w_i is w_i itself, which has fitness $F(t|s)$. Furthermore, the only incumbents in the neighborhood of w_i are v_i and x_i, both of which have fitness $F(s|t)$. By assumption, $F(t|s) > F(s|t)$, thus, under this stronger notion of contraction, neither of the incumbent-mutant edges, (v_i, w_i) and (x_i, w_i), will contract.

Chapter 5

An Experimental Study of the Coloring Problem on Human Subject Networks

5.1 Introduction

Since the pioneering "small-world" experiment [99, 78], there has been a long and fascinating literature examining the structural and navigational properties of natural social networks. Findings range from the now-familiar "six degrees of separation" to more recent theoretical explanations of the heuristics people might employ to exploit social network structure [28, 105, 68, 69, 36]. This theoretical work suggests that structural properties of naturally occurring networks are important in shaping behavior and dynamics. However, the relationships between structure and behavior are difficult to establish in empirical field studies of existing networks. In such studies, the network structure is typically fixed and given which prevents the investigation of how the same group of subjects would behave in alternative networks. A different approach, which we adopt, is to conduct controlled laboratory studies in which network structure is deliberately varied. This provides the first of two main reasons for the work we describe in this chapter.

Much of the previous work in this specific area of social network theory can be summarized in computer science terminology: Using relatively local information, distributed human organizations can collectively compute good approximations to the all-pairs shortest paths problem. Given the volume and visibility of this research, it is perhaps surprising that there is little work on its natural generalization — namely, what *other* types of distributed optimization problems can humans networks solve? Answering this question provides the second main reason for the experiments we conducted and will introduce next.

In this chapter we describe findings from a series of behavioral experiments we have been conducting at the University of Pennsylvania. Human subjects attempt to perform distributed graph coloring in a setting in which each subject controls the color of a single vertex in a large and potentially complex graph. At any point, each player can choose a color for his or her vertex, attempting to select one that differs from the colors of all neighboring vertices. The experimental system allowed us to vary the graph topology, the locality of information given to subjects, and the incentive scheme[1]. Further details on the choices made for each of these experimental design variables are provided in Section 5.2. Our main findings were:

- Distributed human organizations can indeed solve difficult coloring problems. In our experiments, populations of 38 subjects found optimal colorings of 84% of the graphs they were given within 5 minutes, often taking considerably less time to solve them. Among a variety of other challenges, these experiments included graph structures in which the chromatic number was very low despite there being many high-degree individuals.

- Graph topology has strong and systematic effects on behavioral performance. For instance, within a sequence of graphs produced by a "small-world" generative model [103] (which mixes the local connections of a cycle with a variable

[1] As we describe later, subjects were paid according to performance under two different schemes.

number of long-distance chords), we find that average solution time decreases monotonically with graph diameter, despite the fact that the introduction of chords increases the number of coloring constraints. The graphs produced by this generative model also appear to be considerably easier for subjects than those generated by *preferential attachment* [95, 8].

- When the locality of information provided to subjects (which ranged from an entirely local view to the ability to see the current color assignments of the entire graph) was varied, solution time varied in a sometimes surprising way. Within our small-world graphs, more information corresponded to a shorter solution time. However, within the more dense preferential attachment graphs, more information was accompanied by an *increase* in solution time.

- Varying the incentive scheme from a "team" or collective structure, in which individuals are paid only if the entire graph is properly colored, to a "selfish" or individual one, in which individuals are paid as long as they do not participate in any coloring conflicts, corresponded to a mild change in average experiment duration, but was accompanied by a drastic change in cooperative behavior. Under collective incentives, subjects changed color much more often when they have no coloring conflicts, possibly in order to help the overall population escape perceived local minima in solution space.

Along with these primary findings, we also combine statistical analysis of the experimental data with subject's self-reports to propose a natural and simple behavioral model.

For these early experiments, we have chosen to examine a number of experimental design variables simultaneously in order to map out a broad research agenda for future work. Our experiments deliberately blend ideas, methods and problems from sociology, computer science, and behavioral economics. They shed early light on the way human organizations deal with difficult distributed optimization problems:

whether they can solve them, under what conditions, and using what strategies.

5.2 Related Work

It is often thought that structural properties of naturally occurring networks are influential in shaping individual and collective behavior and dynamics. Examples include the popular notion that "hubs or connectors" are inordinately important in the routing of information in social and organizational networks [99, 45, 67]. A long history of research has established the frequent empirical appearance of certain structural properties in networks from many domains, including sociology [99, 78, 47], biology [97, 102], and technology [16]. These properties include small diameter (the "six degrees of separation" phenomenon), local clustering of connectivity [104], and heavy-tailed distributions of connectivity [95, 8, 14]. All of these works study one social network frozen in time, thus it is difficult to study how topology affects the behavior of the individuals in the network. Theoretical models have sought to explain how these structural properties of social networks may interact with network dynamics [68, 69, 36]. This work is the first to give *experimental* evidence as to the relationship between network structure and network dynamics.

There have been a few studies of how social networks change over time. The authors of [6] study how groups form, grow, and evolve in the co-author network and the LiveJournal social network. In [71] the authors study the evolution of social networks by analyzing the e-mail patterns between individuals in a large university over the course of one year. Both of these works give insight as to the dynamics of the behavior in these networks. But, they are unable to systematically modify the topology of these networks making it difficult to understand how the topology affects the behavior of the players.

5.3 System and Experimental Methodology

Our experiments and the system our subjects used to make color choices were both designed to permit the investigation of three main design variables: the structure or topology of the graph being colored; the amount and locality of information given to each subject; and the incentive or payment scheme. Before providing details on the system, our procedures, and the values for these experimental design variables, we describe the graph coloring problem and our reasons for selecting it.

An instance of a graph coloring is simply an undirected network, and the goal is to compute a *proper coloring* of the graph — that is, an assignment of a color to every vertex such that no pair of vertices connected by an edge are assigned the same color. The smallest number of colors required in order to properly color a graph is called its *chromatic number*. This problem has a long history which dates back to the late 19th century [18, 57, 44].

Our reasons for conducting a behavioral study of graph coloring were fourfold. First, we were interested in choosing a problem which had a notably different complexity, from the computer science perspective, than the shortest paths problem, which has been the focus of the long literature that partially inspired this research. While shortest paths is known to be a computationally easy problem for centralized computation, graph coloring is notoriously hard, since it is NP-hard to even weakly approximate the chromatic number [76, 40, 64, 65]. Second, we were interested in a problem that, despite being possibly challenging to solve, was easy for human subjects with no special background to quickly understand. Third, we sought a problem which requires global coordination for its solution, but in which subjects could all locally verify their contribution to or hindrance of this solution. Finally, the graph coloring problem is a natural abstraction of many human and organizational problems in which it is desirable or necessary to distinguish one's behavior from that of neighboring parties. As a specific scenario, consider the problem faced by faculty members scheduling departmental events-recurring classes, one-time seminars, exams, and so on-in a limited number of available rooms. We can view the events to be scheduled as the vertices in a network, with an edge connecting any pair of events that temporally overlap, even partially. Clearly, two such events must be assigned to different rooms or "colors", thus yielding a natural graph coloring problem. Furthermore, even when there is a centralized first-come, first-serve sign-up sheet for rooms, this mechanism is simply the starting point for the negotiation of a solution, and the problem is still solved in a largely distributed fashion by the participants: Faculty members routinely query the current holder of a room whether they might be able to switch to a different room, whether their event will really require their entire time slot, and the like. Other coloring-like problems arise in a variety of social activities (such as selecting a cell phone ringtone that differs from those of family members, friends, and colleagues); technological coordination (selecting a channel unused by nearby parties in a wireless communication network [107, 52]); and individual differentiation within an organization (developing an expertise not duplicated by others nearby). Graph coloring also generalizes many classical scheduling and other resource allocation problems in operations research, logistics, and other fields [44].

5.3.1 System Description

The system we built for our experiments provides subjects with a simple browser-based visual interface allowing asynchronous updating of color choice and a view of the current experiment's state. The system permits us to execute a pre-planned series of graph coloring problems with specified graph topologies and information views (discussed below). During an experiment, each subject sees an interface divided into two panels (see Figure 5.1). The left-hand or *action panel* provides colored buttons that can be used to change the color of the subject's vertex. The right-hand or *information panel* provides varying amounts of information (described further below) about the color choices of other players, but always includes at least the color choice of the subject and the color choices of the subject's immediate neighbors. The information panel, which is continually refreshed, always indicates how many color conflicts there are in the subject's neighborhood, or if there are none. Edges in the graph that have color conflicts are shown as bold lines. In addition, this right-hand panel always includes a "progress bar" at the bottom indicating how many conflicts remain globally.

The system logs fine-grained temporal data on the exact sequence of events in each experiment. This log contains every color-change event, indexed by vertex or subject number, the color selected, and a timestamp with 1-second resolution. The system also administers entry and exit questionnaires to each subject.

5.3.2 Experimental Procedures

We now briefly describe our experimental protocol, which was approved by Penn's Institutional Review Board process. Sessions were held in a laboratory containing 38 workstations, which determined the size of our subject pool for each session. Fifty-five subjects were drawn from a Penn undergraduate computer science class on a related topic with no prerequisites, and were required to attend a preliminary lecture in which they were instructed about the graph coloring problem, the workings of the system, and the specifics of what they would see and how they would be paid. They had all passed a quiz designed to ensure that they understood the graph coloring problem and the workings of the experimental system, and that they knew their fellow subjects, who would be working on the problem with them, also understood everything.

This chapter describes two sessions conducted in January 2006 and consisting of multiple individual coloring experiments. These sessions were preceded by one in September 2005, run under the same protocol but with different graphs and a different subject pool. We viewed this earlier session as both an exploration of the experimental design space, and a large-scale test of the system (which is difficult to

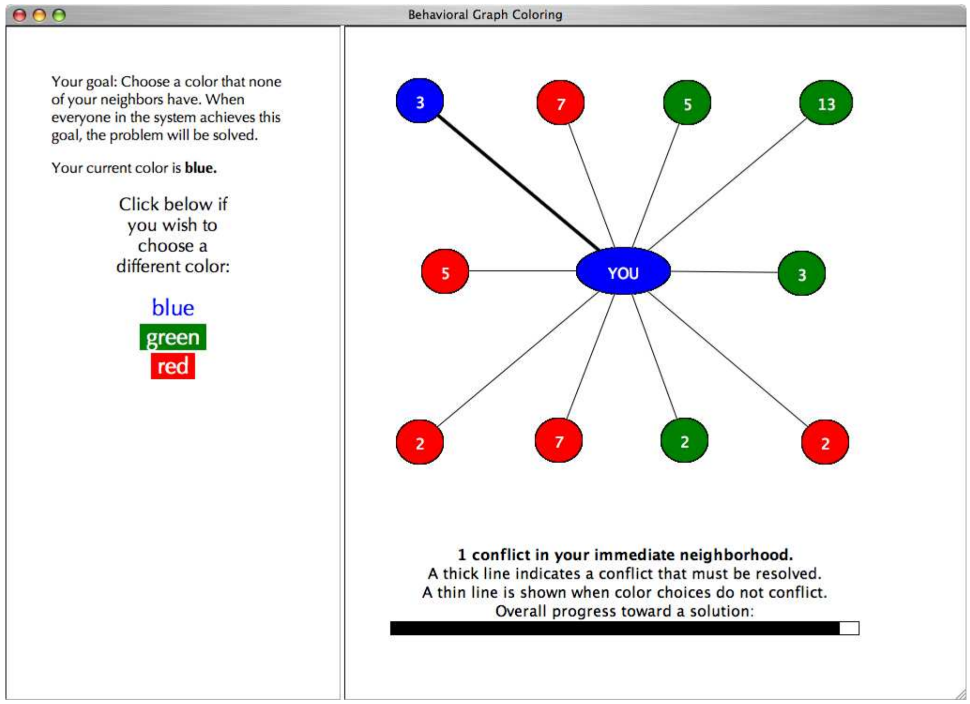

Figure 5.1: A screen shot of the experimental system. On the left-hand side is the action panel, which allows the user to change the color of the node marked "YOU". On the right-hand side is the information panel using the medium information view.

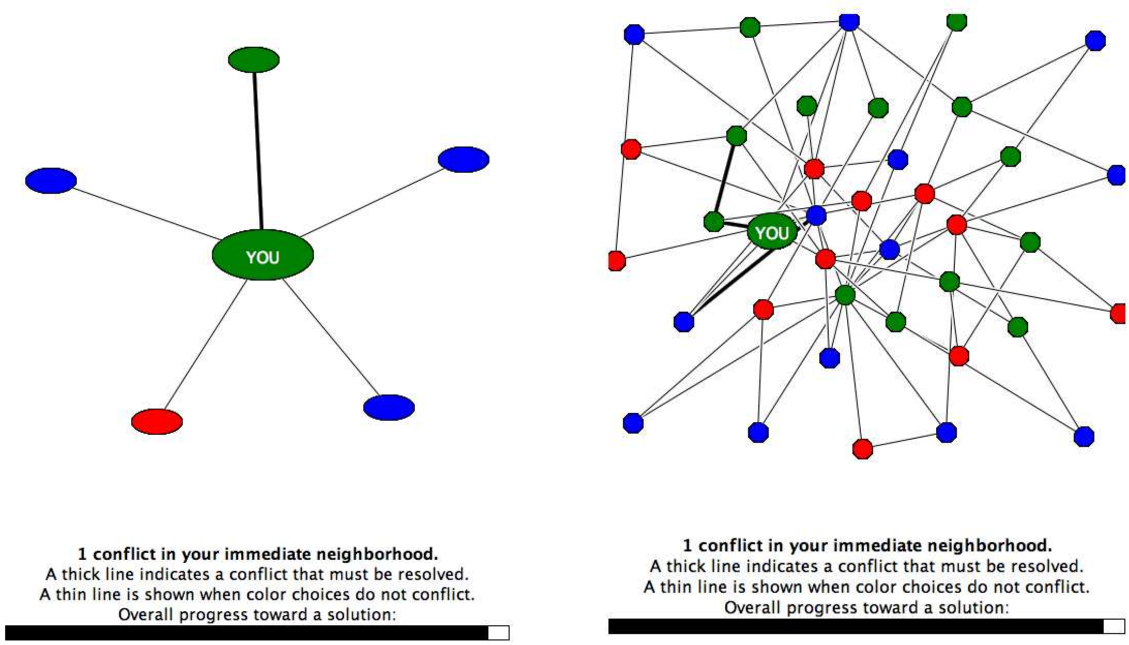

Figure 5.2: Low and high information views as seen by subjects (information panel only). See Figure 5.1 for a full screen shot of the medium information view.

fully test without actually gathering a large number of participants). The September 2005 session was in turn inspired by informal, uncontrolled experiments in graph coloring run in a Spring 2005 Penn course. While both of these preliminary 2005 investigations informed system and experimental design, and gave some indication that humans would be able to solve challenging coloring problems, we report here only on the carefully controlled January 2006 sessions.

Each of the two January experimental sessions consisted of a series of 19 consecutive graph coloring experiments, each with a 5 minute time limit. The sessions lasted between one and two hours. Before the experiments, physical partitions were erected to prevent subjects from glancing at other subjects or their screens. A timekeeper called out how much of the allowed 5 minutes remained. Subjects were carefully observed throughout the session to make sure they were not violating any rules by, for instance, speaking or communicating with any other subjects, attempting to look at the workstations of other subjects, etc. Each problem ended either after 5 minutes or when subjects successfully colored the graph, whichever came first. Then, the session proceeded to the next coloring problem. It is important to note that in each coloring problem, the number of colors provided to subjects was exactly equal to the chromatic number of the graph (which was computed in advance off-line). Thus we deliberately held subjects to the highest standard of optimal coloring, rather than exploring approximations.

By choosing 6 different graph topologies, 3 different information views, and 2 different incentive schemes, we generated a total of $6 \times 3 \times 2 = 36$ unique experimental conditions. All 18 conditions corresponding to one of the incentive schemes were conducted on the evening of January 24, 2006; the 18 corresponding to the other incentive scheme were given the following evening. The order of problems within each session were chosen randomly. In addition, to examine potential "learning" effects, each session ended with an experiment that was identical to the first one of the session, for a total of 19 coloring problems per session. We chose a 5 minute time limit for each of the 19 experiments so the total duration of each of the two sessions, including explanation of the problem and the system, would require less than 2 hours of concentration by the subjects. This was done to minimize the effects of mental fatigue of the participants on our results.

We now proceed to describe our choices of values for our three main experimental design variables, beginning with the graph topologies used.

5.3.3 Choice of Graph Topologies

The space of possible graph topologies is obviously immense. Prior work has shown that people are good at solving the distributed all-pairs shortest path problem in the real world network they are a part of [99, 78, 28]. One of the main motivations behind this work is to discover what other optimizations problems can people solve in this manner. Thus we used models of social networks to generate five of the six different graphs we assigned. However, since it was an open question whether human subjects could solve these kinds of problem efficiently under *any* conditions, we also desired a certain breadth of approach. For these reasons, we drew topologies from two recent but rather different stochastic models for network formation.

The first of these was the so-called "small-world" family, in which a simple cycle is augmented with a variable number of randomly chosen chords. Larger numbers of chords are known to dramatically decrease the (average or worst-case) diameter, and are meant to model long-distance relationships in social networks arising from chance encounters and the like [103]. These long-distance links help in efficiently navigating social networks [69]. To test the effects of these links in the graph coloring problem, we examined three topologies from this family with a varying number of chords: a simple 38-cycle with no additional edges (Simple Cycle), a graph consisting of a cycle with 5 chords (5-Chord Cycle), and a cycle with 20 chords (20-Chord Cycle). Rather than choosing the chords uniformly at random, we selected them at random from among all chords that would not increase the chromatic number beyond the 2

	Chromatic Number	Min. Deg.	Max. Deg.	Avg. Deg.	Std. Dev. Degs.	Diameter (worst)	Diameter (avg.)
Simple Cycle	2	2	2	2	0	19	9.76
5-Chord Cycle	2	2	4	2.26	0.60	12	5.63
20-Chord Cycle	2	2	7	3.05	1.01	7	3.34
Leader Cycle	2	3	19	3.84	3.62	3	2.31
Pref. Att., $\nu = 2$	3	2	13	3.84	2.44	5	2.63
Pref. Att., $\nu = 3$	4	3	22	5.68	4.22	4	2.08

Table 5.1: Statistical properties of each graph

colors required for Simple Cycle[2]. This had the advantage of allowing us to model long-distance connections (and thus reduce diameter) while, in a mathematical sense, making the local appearance of the problem strictly harder (since we have simply added constraints without adding more colors).

The second model we examined is known as *preferential attachment*. In this model, a graph is built incrementally by adding one new vertex at a time. A new vertex is given a fixed number ν of edges to the existing graph; but rather than these edges being chosen uniformly at random, they are directed to an existing vertex with a probability proportional to its current degree. Among other properties, this stochastic model is known to generate heavy-tailed distributions of degree [95, 8, 14] (modeling the social phenomenon of "connectors" [45]) and result in small diameters [7].

Finally, we created one topology in the cycle family intended to represent more "engineered" or hierarchical structures, such as one might find in corporations or the military. In this Leader Cycle, a 36-cycle is augmented by two "leader" vertices, one of which is connected to all even vertices on the cycle, the other to all the odd vertices. The leaders are also connected to each other. The resulting graph remains two-colorable, but has very low diameter and has two vertices with very high degree.

In Figure 5.3 we show all 6 topologies along with a coloring of each. Statistical properties of the 6 graphs are summarized in Table 5.1, where, among other

[2]This amounts to only connecting vertices whose indices have opposite parity.

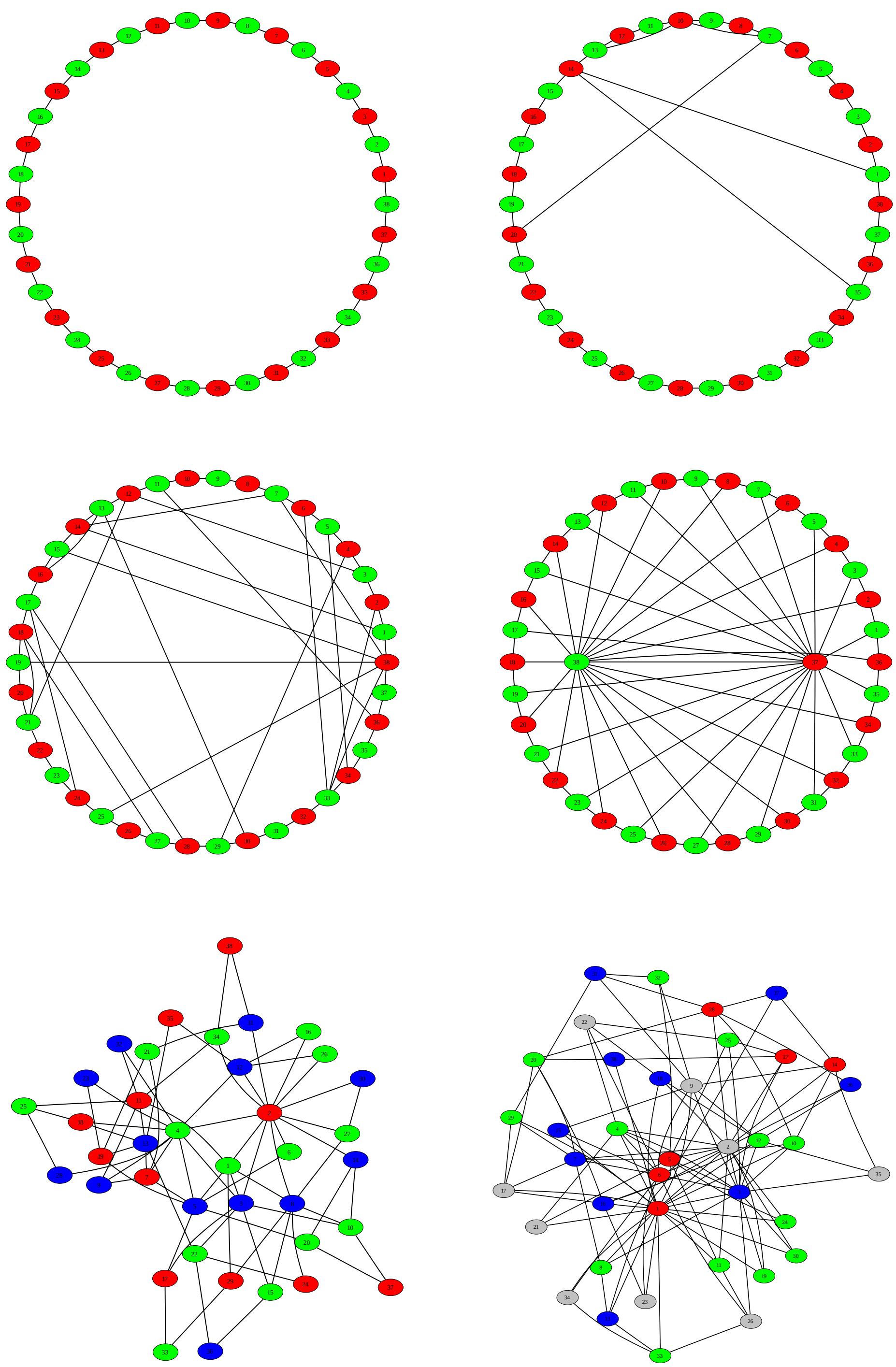

Figure 5.3: Graph topologies with colorings found by subjects. From left to right and top to bottom: Simple Cycle, 5-Chord Cycle, 20-Chord Cycle, Leader Cycle, and Preferential Attachment with $\nu = 2$ and $\nu = 3$.

attributes, we give the worst-case diameter of each graph. This is the distance between the two vertices farthest apart in the graph (the longest shortest path in the graph). The average case diameter, on the other hand, is the average shortest distance between all pairs of vertices (i, j) where $i \neq j$.

Many features of these graphs suggested that they would be challenging for the distributed behavioral protocol we investigated. For instance, in the four cycle-based graphs, every vertex has a degree equal to or higher than the chromatic number of 2, meaning that the local neighborhood has (often far) more players than available colors. Thus, every participant has a coordination problem to solve directly with their neighbors, and indirectly with the population at large. In the preferential attachment graphs, there are a small number of vertices with degree small enough that they will always have a color unused by their neighbors, but the vast majority have rather high degree, and these graphs are the most dense of those we examined. In a similar vein, the preferential attachment $\nu = 2$ graph, which has chromatic number 3, contains 11 complete subgraphs on three vertices (triangles). These triangles form a 14 vertex, connected subgraph of the $\nu = 2$ graph. Since only 3 colors were given, they require 14 participants to precisely coordinate their color choices. Similarly, the $\nu = 3$ graph, which has chromatic number 4, contains 7 complete subgraphs on four vertices (a complete graph on four vertices is denoted K_4). These 7 copies of K_4 form an 11 vertex, connected subgraph of the $\nu = 3$ graph. Since only 4 colors were given for this graph, they require at least 11 people to precisely coordinate their color choices. These multiple subgraphs each represent "embedded" coordination problems that require all the available colors, and must also be integrated into a global solution.

5.3.4 Choice of Information Views

In a particular coloring experiment, our system provided one of three different information views in the right-hand panel of the user interface. As has been mentioned, in each session, we ran three experiments on each graph topology, covering all possible information views of all of the graphs. We also ran each graph and information view combination under both incentive schemes, as discussed below. While the information view varied from problem to problem, in any given exercise *all 38 participants* were given the same view. We have not experimented with different subjects having different views.

In the *low* information view, subjects could see only the color they chose for their own vertices and the colors of their immediate neighbors in the graph. This information view was inspired by theoretical models of search in social networks, such as [69], where each node only has information about its local neighborhood. The *medium* information view is similar, but each neighbor is annotated with the (static) value of its degree. This view was motivated by the desire to provide subjects with some minimal additional information on the local structure that suggested which of their neighbors might have a more difficult coloring task. In the *high* information view, each subject could see the entire graph of 38 vertices as well as the dynamic color choices. We chose this information view to compare the performance of the subjects under the two relatively local views to the global view. This allowed us to understand the effect of restricting the amount of information the subjects were given on their ability to solve the graph coloring problem. We will see that for certain graph topologies the local views helped the subjects arrive at a valid coloring and for other graph topologies it hindered their ability to so. See Figures 5.1 and 5.2 for samples of all three information views. In all three views, the display was continually refreshed to show subject the latest color choices.

5.3.5 Choice of Incentive Schemes

Following the methodology of behavioral economics [17], we paid subjects according to their performance and examined two different schemes for doing so. In the *collective* incentive scheme (which was used on the first of two evenings of experiments),

	Collective Incentive			Individual Incentive		
	Low Info.	Med. Info.	High Info.	Low Info.	Med. Info.	High Info.
Simple Cycle	70	U	26	276	168	25
5-Chord Cycle	134	276, 61	24	221	108	24
20-Chord Cycle	159	39	24	22	39	111
Leader Cycle	24	50	30	74	28	44, 36
Pref. Att., $\nu = 2$	U	U	U	35	83	U
Pref. Att., $\nu = 3$	41	33	U	64	191	U

Table 5.2: Durations for all 38 coloring experiments. Entries denoted U indicate experiments that ended without a successful coloring after the 5-minute (300 second) limit. All other experiments ended with successful, optimal colorings; durations of those are given in seconds. Entries with two values are for the experiments that were repeated twice in each session.

for each of the 19 coloring problems, each subject was paid \$5 for each graph that was properly colored (no coloring conflicts anywhere in the graph within 5 minutes). If even a single conflict remained after 5 minutes, none of the 38 subjects received any payment for that problem. In the *individual* incentive scheme (used the second evening), each subject was paid \$5 if, at the conclusion of a problem (either due to success at coloring the graph or the end of the 5 minutes), *that subject* participated in no color conflicts, regardless of the global outcome.

These two schemes were introduced to allow the study of possible behavioral differences between a “team” and “selfish” incentive mechanism. A natural question to ask is whether such differences can arise in a problem such as coloring, where a subject’s contribution to the global solution is already locally determined. Notable differences in certain measures were in fact seen.

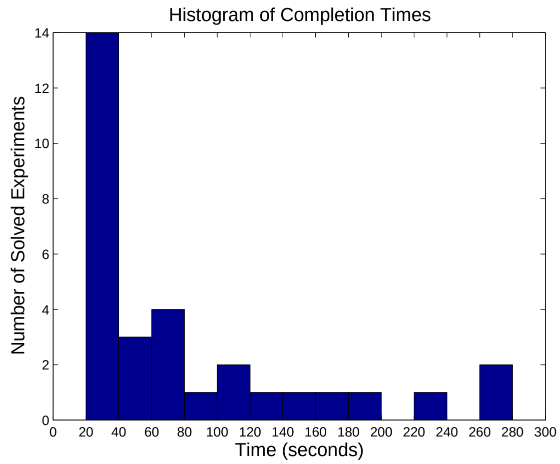

Figure 5.4: Of the 31 solved experiments, 17 were completed in less than 60 seconds.

5.4 Results

Perhaps the most important and surprising of our experimental findings is also the most basic: human populations can solve challenging coloring problems quite well. Of the 38 different coloring experiments we conducted, 31 (82%) resulted in proper optimal colorings within the alloted 5 minutes, and frequently much sooner. The average completion time of the 31 solved problems was only 82 seconds (standard deviation, 75 seconds) and the median just 44 seconds, indicating considerable skew toward low solution times. A histogram of these completion times is shown in Figure 5.4. A complete listing of all 38 experimental durations is provided in Table 5.2.

The number and completion times of the solved problems came as a surprise to both the authors and the experimental subjects. On the exit questionnaire, 52 of the 76 responses indicated that subjects did better than they had expected.

We now examine how performance and behavior varies with each of our three experimental design variables in turn: graph topology, information view, and incentive scheme. Due to the limited number of experiments, for each of these variables we report on averages across some or all variations of the remaining two. Thus, when examining graph topology, for each particular topology of the six, we average

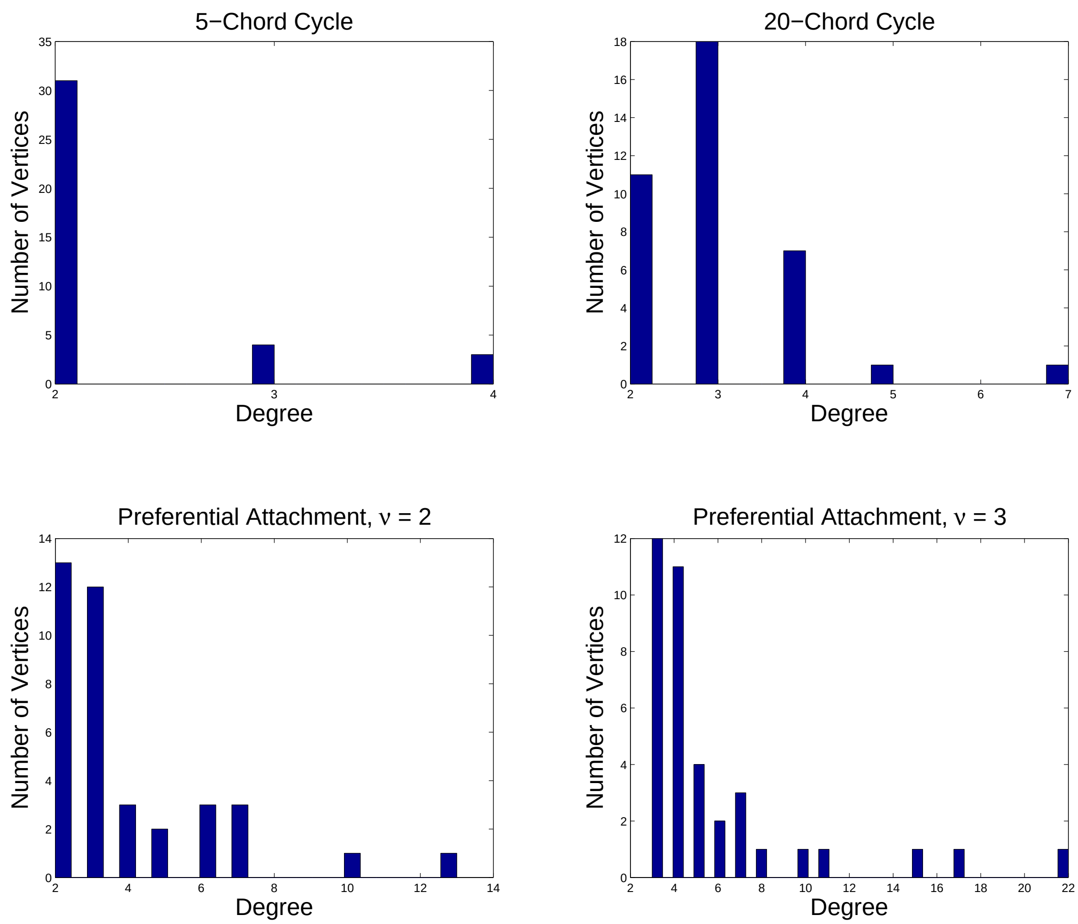

Figure 5.5: Degree distributions: 5-Chord, 20-Chord, Preferential Attachment with $\nu = 2$ and $\nu = 3$.

together all experiments on that topology (which vary information view and incentive scheme); when examining an information view we average across all graphs and incentive schemes run with that view; and so on.

In the subsequent analyses we frequently use average experiment duration as a measure of performance. There is some downward bias in this measure due to the limit (5 minutes) on how long an individual experiment could run, and thus unsolved experiments are included as 300 seconds in the averages. However, the distribution of unsolved experiments was such that allowing these experiments to continue to solution would only have strengthened the results reported here. We also examined alternative measures of difficulty, such as the number of distinct distinct colorings explored and the number of coloring conflicts generated, and found qualitatively similar results. (The correlation between experiment duration and number of distinct colorings explored was: 0.948, between experiment duration and number of conflicts generated: 0.901, and between experiment number of distinct colorings explored and number of conflicts generated: 0.923.)

We will also sometimes choose to report results for the four cycle-based topologies and the two preferential attachment topologies separately due to the rather different motivations and properties of their underlying generative models, and the actual differences in the statistical properties of our specific instances of them (see Table 5.1 and the degree distributions in Figure 5.5).

5.4.1 Effects of Graph Topology

Every one of the six topologies was successfully colored at least twice, under some choice of information view and incentive scheme. The average experiment duration for each graph topology can be seen in the rightmost column of Table 5.3. There are at least two findings of note in these results. The first is that within the family of four cycle-based topologies, there is a monotonic and rapid decrease in duration with decreasing graph diameter, strongly suggesting that smaller diameter results in easier

	Graph Statistics		Experimental Statistics	
	Diameter (worst case)	Diameter (avg. case)	Avg. Experiment Duration (seconds)	Fraction Solved
Simple Cycle	19	9.76	144.17	5/6
5-Chord Cycle	12	5.63	121.14	7/7
20-Chord Cycle	7	3.34	65.67	6/6
Leader Cycle	3	2.31	40.86	7/7
Pref. Att., $\nu = 2$	5	2.63	219.67	2/6
Pref. Att., $\nu = 3$	4	2.08	154.83	4/6

Table 5.3: As the diameter of the cycle-based graphs decreased, so did the average experiment duration. The same trend applied to the preferential attachment graphs.

coloring problems within this family. The second is that preferential attachment graphs appear to be behaviorally much harder than the cycle-based family. We examine each in turn.

The decrease in solution time with diameter[3] for the cycle-based graphs fits with the theory for social navigation or search in "small-world" networks, where random chords decrease shortest paths [103], but here requires a different explanation since the coloring problem is more complex. There is a certain inevitability to the behavioral dynamics on a simple cycle that appears strongly in the data and is worth analyzing since it provides an understanding of the effects of chords and leaders.

Consider a coloring conflict (an edge between two vertices of the same color) occurring on Simple Cycle, where the conflict lies within a larger region that is properly colored. This conflict can be propagated either clockwise or counterclockwise by one of the vertices on the conflict changing color. This moves the conflict to the other edge of the changing player. Two coloring conflicts that "collide" eradicate each other, and such collisions are the only way in which conflicts are resolved. Thus, at a high level, the behavioral dynamics for cycles tend to consist of the propagation of conflicts around the cycle until the experiment concludes. Figure 5.6 visualizes a

[3] The correlation between worst case diameter and experiment duration was 0.466 and the correlation between average case diameter and experiment duration was 0.449.

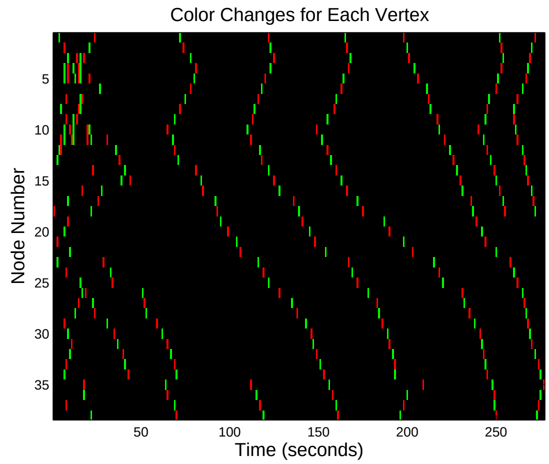

Figure 5.6: Conflict propagation in a simple cycle under the low information view and individual incentives. Here each vertex or player is represented as a row, and each second in the experiment as a column. The colored marks indicate players and times where that color was chosen. The wave-like patterns clearly demonstrate the cyclical conflict propagation discussed in the text. This experiment was successfully solved in 276 seconds.

particularly striking example. The introduction of chords may accelerate these dynamics by increasing opportunities for "distant" (along the cycle) conflicts to collide. They may also serve a coordinating function in that regions on opposite sides of the cycle can now "see" the parity of alternating color being constructed on the other side, a point which we elaborate and support when we examine the effects of information view. We note that the highly regular and centralized structure of Leader Cycle, which deliberately introduces (potentially) coordinating parties, resulted in the lowest average duration.

The second noteworthy finding is that the preferential attachment graphs seem to be considerably harder than the cycle family (see average durations in Table 5.3), with $\nu = 2$ having an average duration much larger than $\nu = 3$. In addition, the duration times for the cycle-based networks and those for the preferential attachment networks passed a two-tailed, unequal variance t test for different means at $P = 0.03$. Most strikingly, six of the 7 unsolved experiments, across all 38, experiments were preferential attachment graphs given under different information and incentive

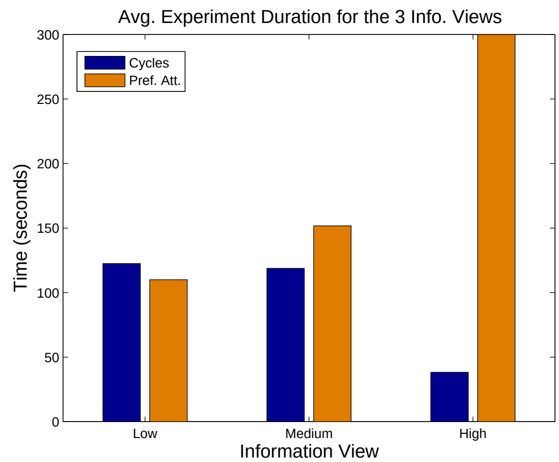

Figure 5.7: More information resulted in a decrease in experiment duration for the four cycle-based graphs, and an increase in duration for the two preferential attachment graphs.

conditions (see Table 5.2).

5.4.2 Effects of Information View

We now turn our attention to the effects of information view. Here our main finding again highlights a distinction between the cycle-based and preferential attachment graphs. While more information is correlated with a decrease in experiment duration for cycle-based graphs, it is accompanied by an increase in the preferential attachment graphs. Figure 5.7 exhibits these two trends, which we now discuss.

In the four cycle-based graphs there are only two proper colorings: one in which even vertices are red and odd vertices are green, and the reverse of this. In the low information view, subjects were unable to see which of the two possible colorings the population as a whole was approaching. This resulted in the collective behavior shown in the left panel of Figure 5.8, where the population oscillated between approaches to each of these two solutions. (This behavior is only slightly reduced in the medium information view.) In the high information view the situation is quite

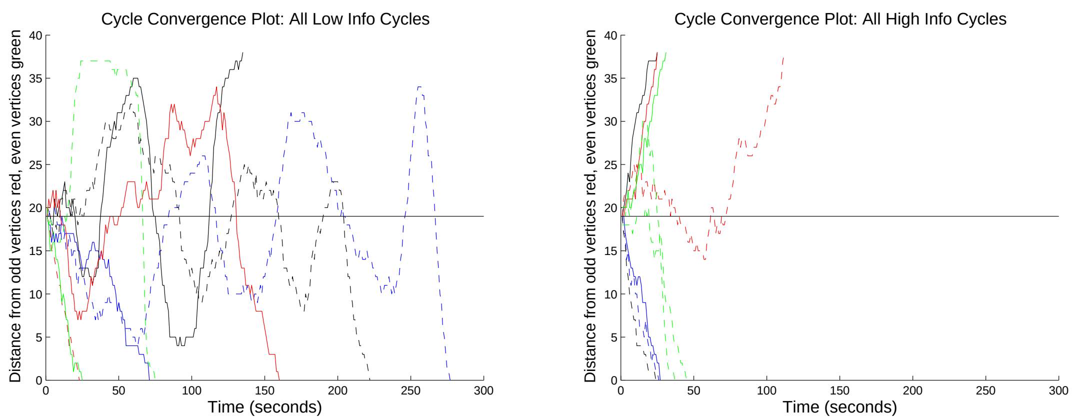

Figure 5.8: Population convergence to one of two possible proper colorings for the four cycle-based graphs. Given a (partial) choice of colors by the population, distance to each of the two solutions was computed as follows. Players whose currently chosen color disagrees with the color of their vertex in the solution count as +1 towards the distance; players that have not yet chosen a color count as +1/2; and players that have chosen the same color as their vertex in the solution count as 0. All experiments begin at distance 19 from both solutions as no player has yet chosen a color. Points below the horizontal line at y=19 are closer to the coloring where odd vertices are red and even vertices are green. A y-axis value of 0 indicates convergence to this coloring. Points above this line are closer to the reverse coloring, and a value of 38 indicates convergence to it. The solid lines represent experiments done under collective incentives, and the dashed lines represent individual incentives. Simple Cycles are shown in blue, 5-Chord Cycles are black, 20-Chord Cycles are red, and Leader Cycles are green.

different, as now each subject could see to which solution the population was converging. The oscillations were not present in this condition, as shown on right panel of Figure 5.8.

In contrast, no such common understanding appears to exist for preferential attachment, and the global information view seems to greatly hamper subjects. All four trials of preferential attachment graphs with global views ended without solution after 5 min. Possible explanations include “information overload” due to the apparent complexity of the networks or their visual layout, combined with the rapid dynamics of the global color selection process. Alternately, it could simply be that time spent by subjects examining the activity in more distant regions of the network

	Average Conflict Duration (seconds)
Low Info.	4.60
Medium Info.	5.21
High Info.	5.56

Table 5.4: On average, conflicts endured for a longer period of time as subjects were given more information.

distracts them from attending to their own local subtask in the global coordination problem, thus slowing collective solution. With further study, such findings may have implications for areas such as information sharing across large organizations and the design of user interfaces for complex systems for multiparty coordination.

Another effect of information view on the behavior of the participants is that conflicts persisted longer when subjects had more information, as shown in Table 5.4. (A conflict remains until one of the two vertices it is incident on changes to a non-conflicting color.) This increase in conflict duration under higher information could be due to subjects being more patient, or it could be that subjects are being distracted by, or are busy processing, the higher-information view that they have of the graph.

5.4.3 Effects of Incentive Scheme

When averaged across the other experimental design variables, incentive scheme appears to have had a rather mild effect on average experiment duration (Table 5.5). Considered as a factor by itself, individual incentives also seemed to reduce conflict duration, perhaps explained by a desire to eliminate local conflicts as quickly as possible. However, any effects from the change in incentive scheme will be conflated with the effects of learning — if there are any — since the individual incentive subjects included many (21 of the 38) who had been present during the previous collective incentive session.

In six of the exit questionnaires, participants wrote to the effect that if they perceived the population to be stuck in a local minimum while searching for a proper

	Avg. Experiment Duration (secs)	Avg. Conflict Duration (secs)	Number of Perturbations
Collective Incentive	130.11	5.39	130
Individual Incentive	113.11	4.87	51

Table 5.5: Changing incentive schemes had a mild effect on average experiment duration and average conflict duration. However, it had a major impact on the amount of cooperative behavior exhibited.

coloring, they would change colors, even if they had no conflicts. We call the event of a player changing colors when they had no conflicts a *perturbation*. Such perturbations are roughly analogous to the introduction of randomization or "thermal noise" into common search algorithms such as simulated annealing. Based on these comments, we counted the number of perturbations for each experiment, leaving out those perturbations where the last time between having no conflicts and changing colors was 2 seconds or less. (Including these short perturbations does not qualitatively change our results, but we have excluded them to account for cognitive reaction time and system lag.) Table 5.5 shows that there was a significant difference in the number of these types of perturbations under the two different incentive schemes. Given the nature of this metric and the incentive schemes, we believe this difference is almost entirely due to the incentive scheme and has very little to do with learning. However, further experimentation will be necessary to test this claim.

5.5 Algorithmic Modeling of Behavior

It is both natural and important to attempt to build simple behavioral models based on the findings we have reported here. Next, we briefly discuss our initial efforts in this direction.

The exit questionnaire administered to all subjects asked them for self-reports on the strategies they used during the experiments. From these self-reports emerged at least two natural and frequently mentioned heuristics:

1. *Choose the color with the fewest local conflicts.*
   (Mentioned on 11 questionnaires.)

2. *Defer to high-degree neighbors.*
   (Mentioned on 39 questionnaires.)

We now describe a simple distributed algorithm incorporating these two heuristics. We describe the algorithm's behavior at a single vertex; the overall algorithm consists of independent copies of this local behavior. If there is currently no conflict between its chosen color and its neighbors' colors, the algorithm does nothing. If there is a conflict, but there is an unused color, the algorithm chooses the unused color. If there is conflict, but all colors are used in the local neighborhood, the algorithm chooses a new color randomly, but in a way that favors colors that are "rare" in the neighborhood.

In the low information view version, "rare" means that the algorithm simply chooses colors with probability inversely proportional to their frequency in the neighborhood. In the medium information view version, "rare" means colors are chosen inversely proportional to their *degree weighted* frequency. The low information version captures the first of the two self-reported heuristics above, while the medium information version captures both. Since few self-reports cited strategies specifically requiring the full information view, we defer the development of heuristics for that condition.

The second and third columns of Table 5.6 show the results of simulations of this distributed algorithm on each of our graphs in both the low and medium information versions described above. We note that this simple algorithm already reproduces some of the behavioral findings — for instance, the general decline of running times within the cycle family with decreasing diameter, including the leader cycle being by far the easiest to solve. However, the behavioral ordering of the 5- and 20-Chord cycles is not captured, nor was the behavioral difficulty of the preferential attachment graphs. We note that the only two orderings of difficulty from the

	Average Number of Changes to Solve		
		(low info)	*(medium info)*
	Uniform	Rare	Rare
Simple Cycle	398.21	396.05	399.60
5-Chord Cycle	687.34	360.95	274.74
20-Chord Cycle	12032.31	381.82	288.38
Leader Cycle	11792.76	215.99	115.30
Pref. Att., $\nu = 2$	667.96	194.41	108.81
Pref. Att., $\nu = 3$	1048.15	215.53	107.63

Table 5.6: From simulation runs ($N = 10000$) of a naive model (uniform) and the proposed model (rare), which prefers to choose colors that are assigned to few others in the neighborhood.

behavioral data that pass tests of statistical significance — namely, that Leader Cycle is easier than 5-Chord Cycle ($p = 0.035$) and Preferential Attachment $\nu = 3$ is easier than 5-Chord Cycle ($p = 0.017$) — are replicated by the model. We also note that a heuristic simply choosing a random color in response to conflicts — with no accounting for neighboring degrees or color frequencies — replicates the behavioral findings considerably more poorly (first column of Table 5.6).

Clearly this is a fledgling modeling effort and there is much more to be done. One avenue we will pursue is the estimation or learning of a detailed state-based behavioral model (in which color changes can be a function of current local color distribution, neighbor degrees, duration of the experiment so far, and a variety of other temporal variables), for which we can exploit the considerable volume of step-by-step, individual color change decisions we have from the experiments.

5.6 Conclusions and Future Work

This chapter has described the first behavioral experiments on distributed graph coloring, a challenging but natural optimization problem, and found that human subjects can perform surprisingly well. Preliminary but evocative findings show strong dependence of behavior and performance on graph structure, information view, and incentives.

Obviously we have examined but a small fraction of even the limited experimental design space we have introduced. A major component of our future work will be to continue to explore this design space, as well as to introduce further dimensions. The human subject networks we studied were small, a perhaps necessary consequence of the carefully controlled, simultaneous play experiments. It is tempting to contemplate Web-based studies [93] on a much larger scale, which will require addressing incentives, attrition, communication, and many other issues. The network topologies examined here were but a sampling of the rich space of possibilities and recent network formation models. Rather than imposing a chosen network structure on subjects, it would also be interesting to consider scenarios in which the subjects themselves participated in the network formation process, while still allowing some variability of structure. Future work should consider an even wider range of natural collective problems and activities. Candidates include problems of agreement or consensus rather than differentiation, and problems involving the formation of local teams or subgroups specifying certain properties (such as being fully connected or having at least one member of each of a fixed number of types or roles). We view the research agenda outlined here to be a behavioral complement to our ongoing theoretical work attempting to relate structural properties of networks to strategic outcomes.

Chapter 6

Conclusion and Future Work

6.1 Conclusion

This thesis examines a variety of different models using a variety of different techniques, but all of the models have a few characteristics in common, and all of the analysis techniques aim to understand the same relationship. Each model examines an economic or game theoretic interaction over a variety of network topologies. Similarly, the various analysis techniques, which include human subject experimentation, simulation with real and artificial data, and theoretical analysis, are all used to understand the effect of network topology on behavior. More specifically, Chapter 5 describes the first controlled, human subject experiment designed to glean how arranging people in different networks impacts their ability to collectively solve a problem. The data from the experiment showed that decreasing the diameter of the networks made it easier for the subjects to find a valid coloring of the network they were arranged in. Chapter 2 analyzes a much more complex model of economic exchange over a significantly richer class of network topologies than previous work. Chapter 3 analyzes a similar model except the formation of the network is considered part of the game. This is the first work that gives such a complete characterization of equilibrium networks for a market based, network formation game. Moreover, the theorems, experiments, and simulations in the models of Chapters 2 and 3 show that the equilibrium wealth of the players can be quantified in terms of the network topology. Finally, Chapter 4 describes a model of evolutionary game theory, which is often used to model social and biological interactions. Again, this is the first time that an evolutionary game theoretic model of such generality was studied over networks. The theorems in this chapter show that the random matching scheme of the classical model of evolutionary game theory is equivalent, for the the purposes of characterizing stable strategies, to randomizing the network or randomizing the mutations.

All of these results begin to answer the central question of this thesis: if players are arranged in a network, and they are strategically interacting only with other players in their local neighborhood, how does the topology of the network affect the outcome of the interaction? Answering this question is important for two reasons. First, it will help the scientific community understand how current networks affect the behavior and dynamics of agents in modern distributed environments. Second, it will help us understand which network topologies are best suited for specific application domains, which will in turn aid in the building of future networks.

These research directions are significant because the rise of the Internet has made the interaction of self-interested agents with possibly conflicting goals much more prevalent. In addition, these types of models can be applied to networks of humans (be they individuals, firms, or organizations) who interact with a similar variety of strategies and goals. These interactions between selfish agents necessitate powerful game theoretic and economic models. Furthermore, for this work to have a broad impact, these models must incorporate network topologies similar to those seen in nature. Fortunately, the burgeoning field of social network theory is producing many studies on the topology of these types of networks. Techniques from computer science can be used to theoretically analyze and experimentally simulate economic models over these types of networks.

The most important, and broadest, contribution of this thesis is the exploration of the intersection of the fields of economics, sociology, and computer science. Work in this area is relatively new, with many open directions that we discuss in the next section. Despite its recent emergence, the work in this thesis shows that many of the theoretical and experimental tools necessary to study this field already exist, such as the behavioral economics paradigm for human experimentation, economic and game theoretic models, generative models from social network theory, and a rich theoretical literature on graph theory. Although surely more techniques will need to be developed. Moreover, given the real world applications, work in the intersection of these fields is certainly well motivated.

6.2 Future Work

This section provides several different avenues for future research which extend the study of the effect of network topology on behavior. In addition they all combine ideas from computer science, game theory, and economics.

6.2.1 Computational Evolutionary Game Theory[1]

Most evolutionary game theoretic models consider an infinite population of agents. These agents usually obey some simple dynamic such as imitation or replication. Typical results in these models show that in the limit (as time goes to infinity) the population converges to an equilibrium. A major open problem in the intersection of evolutionary game theory and theoretical computer science is to analyze a population of n agents, who obey one of these dynamics, and bound the time of convergence to an equilibrium. The notions of equilibrium and stability might have to be adapted to this new finite setting. Results along these lines would yield simple, distributed algorithms that agents could implement and converge to an equilibrium in a bounded

[1] This section was taken from [98].

(and hopefully short) amount of time. This would provide contribution beyond proving the existence of equilibria, and beyond showing that an infinite population will eventually converge to it. It will show that a population of a given size will converge to a stable equilibrium within a certain amount of time.

To start on this endeavor, the simplest models could consider n agents, where each agent could interact with each other agent. One example of such a problem would be to analyze a selfish routing model, such as the one described in [41], except with n agents, as opposed to infinitely many, and show a strongly polynomial time bound for their convergence. After baseline models such as this have been developed and studied, one might then try to find dynamics that result in these agents converging to an equilibrium that maximizes an appropriate notion of social welfare. Another extension would be to consider models where agents are arranged in a graph and can only interact with agents in their local neighborhood. There are many possible graph topologies that could lend themselves to this type of analysis such as dense graphs, random graphs, expanders, and graphs motivated by social network theory. One could then analyze not only the effect of the graph topology on equilibrium, as was done in Chapter 4, but also how it affects the convergence time.

It may turn out that hardness results stand in the way of such progress. Then one could try to bound the time of convergence to an approximate equilibrium, or simply bound the amount of time the population spends far away from an equilibrium. Also results such as the one given in [85], imply that there exist games for which it is hard to compute equilibria. There still could be many well-motivated classes of games for which arriving at an equilibrium is computationally tractable.

6.2.2 Dynamics for Graphical Evolutionarily Stable Strategies

Chapter 4 defines a notion of graphical evolutionarily stable strategy, and the theorems in the chapter exhibit families of graphs for which this definition is equivalent to the classical definition. An interesting open problem in this line of work is to define graphical dynamics that converge to such an equilibrium. Next we provide two examples of dynamics that are inspired by the generative models from social network theory that may lend themselves to this type of analysis. Consider an initial "seed" graph which has a population of incumbents and mutants. At each time step choose a node at random with probability proportional to its fitness. Let u denote this node. One type of dynamic would be to add a new node v of the same type as u, and to attach v to u as well as all the nodes adjacent to u. This would model a parent organism reproducing and the child having the same interactions as the parent. This is similar to the attachment model of [70]. A second type of dynamic would have the new node v attach to the existing graph via the preferential attachment scheme described in Section 2.5. This would model situations where and organisms offspring does not necessarily experience the same interactions as its parent. Furthermore, both of these models may be amenable to analysis with more than just two types of organisms.

6.2.3 Graphical Inequality Aversion

One of the basic assumptions of game theory is that all players are completely rational. The central thesis of behavioral game theory, however, posits that *people* are not always completely rational. This field introduces models that describe and quantify how people deviate from full rationality. Fehr and Schmidt introduce one such model which is called inequality aversion [39]. It was designed by analyzing data from behavioral experiments of the ultimatum game, which we describe next. The ultimatum game is a two-player game, one player is the proposer and one player is the responder. The proposer is initially given \$10, and decides to give x amount to the responder, where $\$0 \leq x \leq \$10$. Then the responder decides to either accept or reject the offer. If the responder accepts, he or she gets $\$x$, and the proposer gets $\$10-x$. If the responder rejects, neither party receives any money. If both players were rational then the proposer would offer \$0.01, and the responder would accept. The results of the behavioral experiment, however, were quite different. Roughly 60%-80% of the offers were in the \$4-\$5 range. Also, only 3% of offers were below \$2 and they were frequently rejected. The model of inequality aversion captures this deviation from rationality by subtracting a "guilt" and "envy" term from the utility functions of both players. In the case of the ultimatum game, the guilt term describes the proposers aversion to earning more than the responder, and the envy term describes the responders aversion to earning less than the proposer.

Analyzing this model in conjunction with the ultimatum game in a general network setting is an open problem [58]. This would provide an excellent first step, however, the ultimatum game may only generate trivial equilibria in a networked setting. Even if it does not, analyzing a more complex bargaining scheme would be interesting in its own right. Corominas-Bosch [25] describes an iterated bargaining mechanism, which is a natural extension of the ultimatum game, in a networked setting (this work is described in more detail in Section 2.2). One plausible result from this line of inquiry could be that for a certain class of games, as the diameter of the network increases, so does the difference between the highest and lowest paid player. Analyzing this type of mechanism, using the model of inequality aversion, over networks motivated by social network theory may yield new insights into wealth variation in the real world.

Inequality aversion also inspires an open problem that is more computational in nature. One can view the guilt and envy terms of the inequality aversion model as a transformation of the payoff bimatrix of a normal form game. It has recently been shown that computing the Nash equilibria of a normal form came is PPAD-complete [19]. Another future direction is to analyze the computational complexity of Nash equilibria of normal form games under the transformation defined by this model. This future direction comes from [58]. Studying models of graphical inequality aversion is another example of a well motivated line of research on the intersection of economics, computer science, and sociology.

6.2.4 The Effect of Network Topology on Collusion

The focus of this thesis is to understand the effect of network topology on the behavior of the agents in the network. An extension of this goal is the study of the effect of network topology on the formation of coalitions. Previous work on cooperative game theory analyzes situations where the action of each agent can affect the payoff of each other agent. Some of this work imposes a graphical communication structure where agents can only communicate, and thereby form coalitions, with agents they are attached to (see [66, 15], both of which are described in Section 1.5). Coalitions in this case would form a partition of the communication network. No prior work has studied the case where each agent can only affect the payoff of the agents they are attached to *and* agents can only join coalitions with agents they are attached to. An interesting future direction would be to understand how the coalitional structure of this type of graphical game would affect the possible equilibria.

Appendix A

Technical Lemmas for Chapter 2

Lemma A.0.1. *Let S_n be a sequence of random variables and let $\mathcal{F}_n$ denote the sigma-field of information up to time n. If $E(|S_{n+1} - S_n||F_n)$ is a.s. summable, then with probability 1, $-\infty < \liminf S_n$ and $\limsup S_n < \infty$.*

Proof. It suffices to show bounded variation, namely that $S'_n := \sum_{k=1}^n |S_{k+1} - S_k|$ is almost surely finite. Let $S''_n := \sum_{k=1}^n E(|S_{k+1} - S_k||F_k)$. Then $E(S'_n I(S''_n < M))$ is bounded above by M, so $\Pr(S'_n > M^2, S''_n < M) < 1/M$. Pick M at least $1/\epsilon$ so that $P(S''_n > M) < \epsilon$. Then $\Pr(S'_n > M^2) < 2\epsilon$. Hence $\Pr(S'_n = \infty) < 2\epsilon$. Since ϵ is arbitrary, $\Pr(S'_n = \infty) = 0$. $\square$

Next we provide the proof of Lemma 2.5.3.

Lemma (2.5.3)**.** *There are constants $c_{\nu,\alpha,p}$ such that for all n, j,*

$$E[Y(n)^p] \leq c_{\nu,\alpha,p} E[Y(n)]^p = O\left(\frac{n}{j}\right)^{p\beta} .$$

Proof. Assume for induction that we have proved this for $p-1$ (Lemma 2.5.1 proves the base case for $p = 1$). Next let $z_n := E[Y(j,n)^{p-1}] = O(n/j)^{(p-1)\beta}$, and let $\mathcal{F}_n$ denote the σ-field of information up to time n. Conditional on $Y(j,n)$ we start with the identity

$$E[Y(j,n+1)^p] \quad = \quad Y(j,n)^p +$$

$$(pY(j,n)^{p-1} + O(Y(j,n)^{p-2}) \cdot \Pr(Y(j,n+1) > Y(j) \mid \mathcal{F}_n)\,.$$

Below, we will verify that the above probability is $\beta Y(j,n)/n + O(Y(j,n)/n^2)$. Assuming this for now, we let y_n denote $E[Y(j,n)^p]$. Take expectations on both sides and divide by y_n to get

$$\begin{aligned} y_n^{-1} y_{n+1} &= 1 + \frac{p\beta}{n} + O\left(\frac{1}{n^2}\right) + O\left(\frac{z_n}{n y_n}\right) \\ &= 1 + \frac{p\beta}{n} + O\left(\frac{j}{n^{1+\beta}}\right)\,. \end{aligned}$$

Taking products now yields

$$y_{n+1} = \prod_{k=j}^{n} 1 + \frac{p\beta}{k} + O\left(\frac{j}{k^{1+\beta}}\right)\,.$$

Taking logarithms gives,

$$\begin{aligned} \log y_{n+1} &= \sum_{k=j}^{n} \log\left(1 + \frac{p\beta}{k} + O\left(\frac{j}{k^{1+\beta}}\right)\right) \\ &= \sum_{k=j}^{n} \frac{p\beta}{k} + O\left(\frac{j}{k^{1+\beta}}\right) \qquad (A.1) \\ &= p\beta \log n - p\beta \log j + O(1)\,. \qquad (A.2) \end{aligned}$$

Equation A.1 comes from the Taylor series expansion of $\log(1+x)$. Equation A.2 uses the estimate $\sum_{i=1}^{n} 1/i = \log n + \gamma + O(1/n)$, where γ is Euler's constant. Thus,

$$y_{n+1} = O\left(\frac{n}{j}\right)^{p\beta}\,.$$

which finishes the induction. It remains to establish the bound on $\Pr(Y(j,n+1) > Y(j,n) \mid \mathcal{F}_n)$.

The value of $\beta Y(j,n)/n$ is incorrect only because there is a chance that some of the ν vertices chosen were duplicates and more samples were required. The mean number of extra samples required is given by

$$E \sum_{k=1}^{n} \left(\frac{Y(k,n)}{(\nu+1)n}\right)^2\,.$$

In the case $p > 2$, the induction hypothesis bounds this by

$$O\left(\sum_{k=1}^{n} n^{2\beta-2} k^{-2\beta}\right)$$

which is $O(1/n)$. The probability of attaching to seller j with one of these extra attachments is therefore $O(Y(j,n)/n^2)$ as required. $\square$

Appendix B

Technical Lemmas for Chapter 3

B.1 Proof of Generalization of Theorem 2.4.1

Next we provide a proof of Lemma 3.5.1, which is a generalization of Theorem 2.4.1.

Lemma (3.5.1)**.** *Let $G = (B, S, E)$ be a bipartite graph such that $|B| = n$ and $|S| = m$. Let $\tau > 0$ be the maximum number such that each element of an exchange equilibrium consumption plan $\{x_{ij}\}$ can be represented as $k\tau$ for an integer k.*[1] *Let $G' = (B', S', E')$ be the τ-balanced graph of G. Then there exists an exchange equilibrium consumption plan for G, where the buyers all earn wealth m/n and the sellers all earn wealth n/m, if and only if G' has a perfect matching.*

Proof. ($\leftarrow$ direction) If G' contains a perfect matching, then an exchange equilibrium consumption plan for G can be defined as follows. For every edge of (b^i_k, s^j_l) of the perfect matching, $b^i_k \in B', s^j_l \in S'$ add τ/m units of cash going from b_i to s_j, and τ/n units of wheat going from s_j to b_i. By construction, every buyer $b_i \in B$ has m/τ copies in G', and since there is a perfect matching in G', the amount of wheat earned by b_i is m/n as desired. Similarly, every seller $s_j \in S$ has n/τ copies in G'

[1]Since the utilities and endowments of the players are rational, the values of the consumption plan are also rational [27]. Thus, such a τ must exist.

and since there is a perfect matching in G', the amount of cash earned by s_j is n/m as desired.

($\rightarrow$ direction) If there is an exchange equilibrium consumption plan where all buyers earn wealth m/n for G, then each edge (b_i, s_j), $b_i \in B, s_j \in S$ with $k\tau$ units of cash going from b_i to s_j, can be partitioned into km distinct edges between the sets of nodes $\{b_k^i\}_{k=1}^{m/\tau} \subseteq B'$ and $\{s_l^j\}_{l=1}^{n/\tau} \subseteq S'$. These edges in G' can be viewed as carrying τ/m units of cash from a b_k^i to a s_l^j and they will form the perfect matching edges. Now let us count how many edges each node $b_i \in B$ induces. By the market clearing condition, the total expenditure of b_i is 1 unit of cash. Thus b_i induces m/τ edges which equals the number of corresponding $b_k^i \in B'$.

For $s_j \in S$ we have that its incoming flow is n/m units of cash. Furthermore, every edge incident on s_j was split into km copies each carrying τ/m units of cash from b_k^i to s_l^j. Thus there must be n/τ edges incident on the $\{s_l^j\}_{l=1}^{n/\tau}$. Therefore, each edge can be matched with a unique pair of nodes in G' and form a perfect matching. $\square$

B.2 Lemmas for Theorem 3.4.2

Lemma B.2.1. *Let $C = (\tilde{B}, \tilde{S})$ be an (m, k)-trading component of a bipartite exchange economy G. If*

$$\beta = \underset{B':B' \subset \tilde{B} \text{ and } S'=N(B')}{\operatorname{argmax}} \frac{|B'|}{|S'|},$$

then $\beta \leq \frac{m-1}{k}$.

Proof. Fix a set $B' \subset \tilde{B}$ of size $m-1$. Assume for the sake of contradiction that $|N(B')| < k$, that is $|N(B')| = k + l$, where $l > 0$. Then, $\frac{m-1}{k-l} \geq \frac{m-1}{k-1} > \frac{m}{k}$, which by Corollary 3.5.1 contradicts the fact that C is an (m, k) trading component. Thus $|N(B')| = k$. $\square$

B.3 Construction of Equilibrium Graphs

B.3.1 Exploitation(k, ℓ) graphs

In this section we show that there are Exploitation(k, ℓ) which are Nash equilibria of the network formation game. We start by providing a technical lemma.

Lemma B.3.1. *If $\alpha > 1 - 1/\ell$ then at Nash equilibrium of the network formation game, no seller in a Exploitation(k, ℓ) graph would buy an edge to a buyer of degree ℓ or $\ell + 1$.*

Proof. Let s be a seller, and let b be a buyer of degree ℓ or $\ell+1$. At Nash equilibrium of the network formation game, b's wealth is at most $1/\ell$. Since the players are rational, the only way for trade to occur over a (b, s) edge is if s offered a price lower than $1/\ell$. Since $\alpha > 1 - 1/\ell$, s would only decrease its utility by buying an edge to b. $\square$

Now we are ready to show that Exploitation(k, ℓ) graphs can be equilibria graphs of the network formation game.

Lemma B.3.2. *If $\alpha > 1 - 2/(\max(k + 1, \ell + 1))^2$, then any Exploitation$(k, \ell)$ graph where nodes with degree k or $k+1$ and ℓ or $\ell+1$ buy all the edges incident on them, is a Nash equilibrium of the network formation game.*

Proof. Let s be a seller with degree k or $k + 1$. By Lemma B.3.1 we know that at Nash equilibrium of the network formation game s will not buy any edges to buyers of degree ℓ or $\ell + 1$. Next, let $w_1, \ldots, w_{k'}$ be the buyers attached to s in a Exploitation(k, ℓ) graph. Now say s bought a set of edges that did not contain all of the w_i. Then, since $\alpha < 1$ and no other players are connected to the w_i, s could increase its utility by buying edges to the those unconnected w_i. Thus we can assume, at Nash equilibrium of the network formation, s buys all the edges to the w_i. Next, we show at formation equilibrium s does not buy edges to other degree 1 nodes.

By the market clearing condition, the wealth of each of the w_i is either $1/k$ or $1/(k+1)$, and s wealth is either k or $k+1$. If s has wealth $k+1$ and it buys edges to buyers that have wealth k, these buyers have no incentive to switch to s so it would not be rational for s to buy an edge to such a buyer. Next, if s wealth is k and it buys an edges to a buyer b that is also getting price k, b would not buy from s for the following reason. Assume that if (b, s) is not an edge in the graph b buys from s', but when (b, s) is an edge in the graph b buys from s. Then at market clearing s would offer price $k + 1$ and then s' would offer price $k - 1$. Thus it would not be rational for b to switch sellers. Finally, if s wealth is $k + 1$ and a buyer b is getting price k from s', s will not buy the edge to b. If s did, then b will split its good evenly between s and s'. Since the cost of an edge is $\alpha > 1 - 1/(\max(k + 1, \ell + 1))^2$, where $k, \ell > 1$ and this edge only increased the utility of s by $1/2$, s would not buy this edge.

Next, let s be a seller of degree 1. Again, by Lemma B.3.1 we know that at equilibrium s will not buy any edges to buyers of degree ℓ or $\ell + 1$. So, all we have to show is that at equilibrium s will not buy any edges to buyers of degree 1. Consider the result of s buying a edges to a set of buyers B, where $|B| = m$. If $m > \lfloor(\ell + 1)/2\rfloor$, s wealth would only be $\lfloor(\ell + 1)/2\rfloor$. So $m \leq \lfloor(\ell + 1)/2\rfloor$, in which case s wealth would be m and pay $m\alpha$ for the edges to B. Observe that $m(1 - \alpha) \leq (\ell + 1)(1 - \alpha)/2 < 1/(k + 1)$. Thus, buying these m edges would only decrease the utility of s.

Thus, we have shown that at Nash equilibrium of the network formation game sellers would buy only those edges designated by the Exploitation(k, l) graph. The case for buyers is entirely symmetric. □

B.3.2 Balanced$(k, k + 1)$ Graphs

In this subsection we show that balanced graph are equilibria graphs of the network formation game for appropriate values of α. We start by showing that any $(k, k+1)$

minimal trading component can be part of a balanced graph. Note that in comparison to the previous trading components a $(k, k+1)$ components can differ from each other. We start by characterizing every minimal $(k, k+1)$ trading component.

Lemma B.3.3. *Let $C = (\tilde{B}, \tilde{S})$ be a $(k, k+1)$ minimal trading component then the degree of each $b \in \tilde{B}$ is exactly 2.*

Proof. Suppose, for the sake of contradiction, that there exists a node $b \in \tilde{B}$ with degree 3 in C. Let s_1, s_2, s_3 be its neighbors in $\tilde{S}$. By the minimality of the trading component we have that there exists three subsets S_1, S_2, S_3, such that $s_i \in S_i$ and that $|S_i| - 1 = |N(S_i) \setminus \{b\}|$ and that for every $S' \subset S_i$ this equality does not hold (if such do not exist then we can remove the edge and the trading component will not be effected, which is a contradiction to its minimality). Let $\bar{S}$ be the union of S_1, S_2 and S_3 excluding $\{s_1, s_2, s_3\}$. Since C is trading component, then by Corollary 3.5.1 for every subset S' of $\bar{S}$, we have $|S'| \leq |N(S')|$, and thus there exists a perfect matching between $\bar{S}$ and $N(\bar{S})$, and their cardinality is identical, denote it by ℓ. Now consider the set $\bar{S}$ with $\{s_1, s_2, s_3\}$, its cardinality is $\ell + 3$ however, the cardinality of $|N(\bar{S} \bigcup \{s_1, s_2, s_3\})|$ is $\ell + 1$. This is due to fact that if s_1 for instance will add two nodes to $N(\bar{S})$, then we will have $|S_1 \setminus \{s_1\}| = |N(S_1 \setminus \{s_1\})| + 2$; this yields a higher ratio than C's ratio, which contradicts the fact that C is a trading component by Corollary 3.5.1. Thus the degree of b at most 2. Note that the degree of b cannot be 1 as its wealth is strictly larger than 1. $\square$

Using this characterization for every $(k, k+1)$ minimal trading component, we can show that balanced graphs are Nash equilibria of the network formation game for specific values of α.

Lemma B.3.4. *If $1/(k+1) \leq \alpha \leq 1/k$ then any balanced graph consisting n_1-$(k, k+1)$ trading components and n_1-$(k+1, k)$ trading components is an equilibrium graph of the network formation game.*

Proof. (sketch) Let us consider the strategy of players in a $(k, k+1)$ connected component, C, where $\tilde{B}$ is the buyer set and its cardinality is k and $\tilde{S}$ is the seller set with cardinality $k+1$. Before going over all possible cases, we provide the following fact:

(A) For a graph G, if the lowest wealth obtained by buyers (sellers) is β, then for a graph G', such that $G \subseteq G'$, the wealth of any seller(buyer) is at most $1/\beta$.

We first consider every possible deviation of $b \in \tilde{B}$, whose current utility is $\frac{k+1}{k} - 2\alpha$ by Lemma B.3.3:

1. **Removing one edge:** By the minimality of the trading component, if b removes an edge to s then there exists a subset $S' \subset \tilde{S}$, which now satisfies $|S'| = |N(S')| - 1$. If Algorithm 1 is input $G \setminus \{(b, s)\}$ then S' is removed first. Furthermore, every other subset W of $\tilde{S}$ obeys $|W| \leq |N(W)|$, and thus there is no wealth variation and b's new wealth will be 1. Therefore b gains α from removing the edge but its wealth decreases by $1/k$ and will have no incentive to deviate in this manner.

2. **Removing two edges:** By Lemma B.3.3, b has exactly two edges, and the removal of both will make its utility 0, which is clearly less than its previous utility $\frac{k+1}{k} - 2\alpha$.

3. **Buying additional edges:** By fact (A), b cannot have wealth larger $\frac{k+1}{k}$ by adding edges to G.

4. **Removing one edge and buying additional edges:** By case 1, after removing one edge the wealth of b is 1. Thus the trading components (without the additional edges) consist of: an $(\ell, \ell+1)$-component where $\ell < k$, a perfect matching component of size $(k-\ell)$ (both due to the decomposition of C), and the rest are the $(k, k+1)$ and $(k+1, k)$ trading components. Adding edges to any of $(k, k+1)$ or $(k+1, k)$-components cannot yield a wealth higher than $\frac{k+1}{k}$ as $\frac{k}{k+1}$ is the lowest wealth before doing so, and thus the b has no incentive to

buy one. Now it is easy to see that buying an edge into the $(\ell, \ell+1)$ will form the $(k, k+1)$ trading component again, and buying additional edges (to either the $(\ell, \ell+1)$ component or to the perfect matching component) will have no influence on the prices. Therefore, the node's wealth will be at most $\frac{k+1}{k}$ and it will by at least two edges, and thus will have no incentive to deviate.

5. **Removing two edges and buying additional edges:** By Lemma B.3.3 and the minimality of the trading component, it is not hard to see that after removing two edges, C is decomposed into an isolated vertex b, and (ℓ_1, ℓ_1+1) and (ℓ_2, ℓ_2+1) trading components (note that ℓ_2 can be 0 and that $\ell_1 + \ell_2 = k-1$). The other trading components are the former $(k, k+1)$, $(k+1, k)$ trading components. Once again buying edges into the $(k, k+1)$, $(k+1, k)$ trading components cannot help as the maximal price that can that be obtained is $\frac{k+1}{k}$ and at least two edges must be bought to achieve it. Now buying one or two edges to the smaller components (that are created from the disconnection) is identical to the previous cases.

Now let us consider the possible deviations of a node $s \in \tilde{S}$. Note that current strategy of s is not buying any edges and thus its utility is $\frac{k}{k+1}$. We first note that forming edges to nodes b with current wealth larger than 1 is never beneficial as b will prefer not to trade with s, and such edges will not effect the trading components.

1. **Buying a single edge:** The edge must be to a node with wealth $\frac{k}{k+1}$, and by similar arguments to Theorem 3.4.1, the wealth of s will be 1. Thus the new utility of s would be $1 - \alpha \leq 1 - \frac{1}{k+1} = \frac{k}{k+1}$, which is less than its current utility.

2. **Buying 2 edges:** In such case one can see that s will now be part of a $(k+2, k+1)$ trading component (as a node with wealth higher than 1), however its utility will be $\frac{k+2}{k+1} - 2\alpha$ which is smaller than $\frac{k}{k+1}$ for $\alpha \geq 1/(k+1)$.

3. **Buying at least three edges:** By fact (A) the wealth of s after buying such edges is bounded by $\frac{k+1}{k}$, and thus the increase in its wealth is at most $\frac{k+1}{k} - \frac{k}{k+1}$, which is at most $2/k$. The edges' cost on the other hand is $\ell\alpha$, which is at least $3/(k+1)$, which is always larger than $2/k$. $\square$

The proof of the following lemma is a straightforward generalization of the previous. Thus we omit it's proof.

Lemma B.3.5. *If $\alpha = 1/(k+1)$, then balanced graph consisting n_1 $(k-1,k)$ or $(k,k+1)$ trading components and n_1 $(k-1,k)$ or $(k,k+1)$ trading components is an equilibrium graph of the network formation game.*

B.4 Proof of Theorem 3.4.4

Theorem (3.4.4)**.** *Let G be a Nash equilibrium graph of the network formation game. Then G is equal to its minimal exchange subgraph.*

Proof. Let G' be the exchange subgraph of G. Assume for the sake of contradiction that G' is a strict subgraph of G. There are two types of edges which can be in G and not in G'. The first type consists of edges inside a trading component of G, and the second type consists of edges connecting two trading components of G. Now consider running Algorithm 1 on G.

We first deal with the redundant edges inside a trading component of G. Let C_i be the first connected component output by Algorithm 1 on G such that there exists an edge (u,v) where $u \in U_i$, $v \in N(U_i)$ and $(u,v) \notin G'$. For any strict subset $W \subset U_i$, the edge (u,v) could only increase $|N(W)|$ thereby decreasing $|U_i|/|N(U_i)|$. So if (u,v) were not an edge in the input, Algorithm 1 would not choose any subset of U_i before choosing U_i. In addition (u,v) does not change $|N(U_i)|$ or $|U_i|/|N(U_i)|$. Thus Algorithm 1 would still output $(U_i, N(U_i))$ if (u,v) were not in G. Thus (u,v) can be removed without affecting the exchange subgraph. Since the price of an edge is positive G cannot be an equilibrium graph of the network formation game.

After dealing with edges of the first type, we proceed to the second type. By definition there are no edges joining nodes in U_i to nodes outside the set $N(U_i)$, so we only have to deal with edges that join nodes inside $N(U_i)$ to nodes outside U_i. By Lemma 3.5.3 we have that for every subset of buyers S in G_{i+1},[2] $|S|/|N(S)| \leq |U_i|/|N(U_i)|$. Adding the edges that were removed while moving to G_{i+1}, imply that in G_i, $|S|/|N(S)|$ could only be only smaller (as $N(S)$ might be larger). Thus U_i remains the set that maximizes $|U_i|/|N(U_i)|$ in G_i. So the edges that join nodes inside $N(U_i)$ to nodes outside U_i do not appear in G_j, where $j \geq i+1$. Thus they can be removed without affecting the exchange subgraph. Since the price of an edge is positive G cannot be an equilibrium graph of the network formation game. $\square$

[2] The subgraphs G_i and G_{i+1} of G are defined by Algorithm 1 in Section 3.5.2

Appendix C

Technical Lemmas for Chapter 4

Lemma C.0.1. *If* $p = \Omega(1/n^c)$ *for any constant* $0 \leq c < 1$*, then the minimum degree of a vertex in almost every* G_p *is at least* $(1-\gamma)pn$*, for all constants* $\gamma > 0$*.*

Proof. Fix a vertex $v \in V$, by the Chernoff bound,

$$\Pr\{\deg(v) < (1-\gamma)np\} < \frac{1}{e^{np\gamma^2/2}}.$$

By the union bound (Boole's inequality) we get

$$\Pr\{\cup_{v \in V} \deg(v) < (1-\gamma)np\} < \frac{n}{e^{np\gamma^2/2}}.$$

Since $p = \Omega(1/n^c)$,

$$\lim_{n\to\infty} \frac{n}{e^{np\gamma^2/2}} = \frac{n}{e^{\Omega(n^{1-c})\gamma^2/2}} = 0.$$

Thus for any $\epsilon > 0$, there exists an n_0 such that for all $n > n_0$,

$$\Pr\{\cup_{v \in V} \deg(v) < (1-\gamma)np\} < \epsilon.$$

Taking the complement of this event, we conclude

$$\Pr\{\cap_{v \in V} \deg(v) \geq (1-\gamma)np\} > 1-\epsilon.$$

□

Lemma C.0.2. *If* $c_1, c_2, \ldots, c_m$ *are integers greater than 1 and* $A = \sum_{k=1}^{m} \binom{c_k}{2}$ *and* $B = \sum_{k=1}^{m}(c_k - 1)$ *then* $B \geq \sqrt{A}/2$.

Proof. The proof is by induction on m. For the base case, setting $2(c_1 - 1) \geq \binom{c_1}{2}^{1/2}$ and simplifying results in $7c_1^2 - 7c_1 + 8 \geq 0$. This polynomial is positive when $c_1 \geq 2$. We assume the claim is true for upto m integers.

$$\begin{aligned}
\sum_{k=1}^{m+1}(c_k - 1) &= \sum_{k=1}^{m}(c_k - 1) + (c_{m+1} - 1) \\
&\geq \frac{\sqrt{\sum_{k=1}^{m} \binom{c_k}{2}}}{2} + \frac{\sqrt{\binom{c_{m+1}}{2}}}{2} \\
&\geq \frac{\sqrt{\sum_{k=1}^{m+1} \binom{c_k}{2}}}{2}
\end{aligned}$$

The second line follows from the first line, by the induction hypothesis and by the same argument as the base case. The third line follows from the second line from the inequality $\sqrt{x} + \sqrt{y} \geq \sqrt{x+y}$. □

Lemma C.0.3. *Given a family of graphs* $G = \{G_n = (V_n, E_n)\}_{n=0}^{\infty}$, *and a mutant family* $M = \{M_n\}_{n=0}^{\infty}$, *which is determined by labeling each vertex a mutant with probability* $\epsilon > 0$, *let* $\{\epsilon_n\}_{n=0}^{\infty}$ *be a sequence of random variables such that* $\epsilon_n n = |M_n|$ *for all* n. *Then the sequence random variables* $\{\epsilon_n\}_{n=0}^{\infty}$ *converge to* ϵ *in probability.*

Proof. By Chernoff we get the following two bounds, for all $\tau > 0$

$$\Pr(\epsilon_n > (1+\tau)\epsilon) = \Pr(\epsilon_n n > (1+\tau)\epsilon n) \leq \exp(-\frac{\epsilon n \tau^2}{3}),$$

$$\Pr(\epsilon_n < (1-\tau)\epsilon) = \Pr(\epsilon_n n < (1-\tau)\epsilon n) \leq \exp(-\frac{\epsilon n \tau^2}{2}).$$

Combining these with the union bound shows,

$$\Pr(\epsilon_n \notin (1 \pm \tau)\epsilon) = 2\exp(-\frac{\epsilon n \tau^2}{3}).$$

Thus, for all $\tau > 0$, $\lim_{n\to\infty} \Pr(|\epsilon_n - \epsilon| \geq \tau) = 0$. □

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